Selected problems

VD2. Given quadrilaterals $ABCD$ and $A'B'C'D'$ such that $|AB|=|A'B'|$, $|BC|=|B'C'|$, $|CD|=|C'D'|$, $|AD|=|A'D'|$ and $AC\perp BD$, prove that $A'C'\perp B'D'$.

VD3. Given five vectors on the plane, prove that one can select two of them so that the length of their sum is less or equal to the length of the sum of the three other vectors.

VD4. Let $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ be plane vectors with the length of each not greater then 1, prove that one can select signs in the sum $\vec{c}=\vec{a}_1 \pm \vec{a}_2 \pm \dots \pm \vec{a}_n$ so that $|\vec{c}| \le \sqrt{2}$.

GI2. A square of the size 1 is cut into rectangles. One chooses a shorter side in each rectangle (if a rectangle is a square, one chooses any side). Prove that the sum of the lengths of the chosen sides is not less then 1.

GI3. a) Max and Lessy have each a copy of the same table 5×5 filled with 25 distinct real numbers. Max finds the maximal number M_1 in his table and crosses out the row and the column containing M_1, then he find the maximal number M_2 among the rest and crosses out the row and the column containing M_2, and so on. Lessy do the same with her copy of the table, but each time she finds the least of the numbers L_1, L_2, L_3, and so on. Prove that

$$M_1+M_2+M_3+M_4+M_5 \ge L_1+L_2+L_3+L_4+L_5$$

b) Can it happen that $M_1+M_2+M_3+M_4+M_5 = L_1+L_2+L_3+L_4+L_5$?

FE1. RNO1998 Find every $a\in\mathbb{R}$ such that there exists a non-constant function $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(a(x+y))=f(x)+f(y)$.

DM5. Given n+1 non-empty subsets of the set {1, 2, …, n}, prove that one can mark each subset as either white or black or gray so that the union of the white subsets be equal to the union of the black subsets.

SI3. 109199 Given $x_1, x_2, \dots, x_k > 0$, $x_1^2+\dots+x_k^2 < \dfrac{x_1+\dots+x_k}{2} < \dfrac{x_1^3+\dots+x_k^3}{4}$.

a) Prove that k>50.

b) Give an example of these numbers for some k.

RD3. IMC7-2-4. Let n be an odd positive integer and A be nxn-matrix with entries a_{ij}=2 for i=j, a_{ij}=1 for $i-j=\pm 2 (mod\ n)$ and a_{ij}=0 otherwise. Find det A.

RD4. IMC5-2-3. In the linear space of all real $n\times n$ matrices, find the maximum possible dimension of a linear subspace V such that $\forall X, Y\in V$ $\text{trace}(XY)=0$

(The trace of a matrix is the sum of the diagonal entries.)

ZP4. How many coefficients of the polynomial $(x+1)^{100}$ are even?

MA1. Let n be an odd integer and S be a symmetric nxn matrix over $\mathbb{Z}_2$ with zeroes on the main diagonal. Prove that S is singular.

MA5. IMC16.1.2. Let k and n be positive integers. A sequence (A_1, …, A_k) of $n\times n$ real matrices is *good* if $A_i^2\neq 0$ $\forall i=1,\dots,k$ but $A_iA_j=0$ for $1\le i, j\le k, i\neq j$. Show that $k\le n$ in all good sequences, and give an example of a good sequence with k = n for each n.

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