Continuous Functions

All functions below be $f:\mathbb{R}\to\mathbb{R}$ unless otherwise stated.

Each elementary function is continuous on its domain. Addition, subtraction, multiplication, division by nonzero and composition of continuous functions give continuous functions.

1. *f(lnx)* is continuous for all $x>0$. Does *f(x)* be continuous for all *x*?
2. a) Both $f(x-1)+7f(x+1)$ and $f(x+1)+7f(x-1)$ be continuous. Is *f(x)* continuous?
b) For every real $a>1$, the function $f(x)+f(ax)$ be continuous, prove that *f* is continuous.

Intermediate value theorem. *If the real-valued function f is continuous on the closed interval [a, b] and k is some number between f(a) and f(b), then there is some number c in [a, b] such that f(c) = k.*

3. Let $f(x)$ be continuous on $[a, b]$ and $f(a)f(b)\le 0$. Prove that $f(x)$ has a root on $[a, b]$.

4. Prove that some chord cut from a circle exactly 1/3 of its area.

5. $f(x+1)f(x)+f(x+1)+1=0 \quad \forall x$. Prove that *f* is discontinuous.

6. Given $x_1, x_2, \ldots, x_n \in [0,2]$, prove that $|x-x_1|+|x-x_2|+\ldots+|x-x_n|=n$ for some $x\in[0,2]$.

7. Let $f(x)$ be continuous on [0, 100] and f(0)=f(100). Prove that $\exists a\in[0,\ 99]: f(a)=f(a+1)$.

Extreme value theorem. *If the real-valued function f is continuous on the closed interval [a,b], then the function attains its maximum, i.e. there exists* $c\in[a,b]$ *with* $f(c)\ge f(x) \quad \forall x\in[a,b]$ *. The same is true of the minimum of f.*

8. Let *f* be continuous and $f(x^2)-f^2(x)\ge 0.25 \quad \forall x$. Does that imply *f* has an extremum point?

9. Does there exist a continious function $f:\mathbb{R}\to\mathbb{R}$ that attains each real value exactly 3 times?

10. Prove that $\forall a_1, b_1, a_2, b_2, \ldots, a_n, b_n \in \mathbb{R}$ the equation
$a_1\sin x+b_1\cos x+a_2\sin 2x+b_2\cos 2x+\ldots+a_n\sin nx+b_n\cos nx=0$ has a solution.

11. Given $f:\mathbb{R}_{pos}\to\mathbb{R}_{pos}$ such that $f(x^y)=f(x)^{f(y)} \quad \forall x,y\in\mathbb{R}_{pos}$,

a) find all such continuous functions, **b)** find all such functions.

Credit problems

CF1. (IMC6.2.2) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for any real numbers $a<b$, the image $f([a,b])$ is a closed interval of length $b-a$.

CF2. (IMC8.1.1) Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that
$f(x+q)-f(x)\in\mathbb{Q} \quad \forall x\in\mathbb{R}, q\in\mathbb{Q}$.

www.ashap.info/Uroki/eng/NYUAD18/index.html