Infinite Algorithms

An algorithm can check an infinite sequence of hypotheses one by one until find the right one.

1. An unknown number of rooms are connected by corridors to a ring. You are placed in one of the rooms and have a chalk. You can go at both directions and make marks on the walls as you wish. Sooner or later you should say, how many rooms are in the ring. But you have only one guess. If you are right, you'll be free. If you are wrong, you'll be killed. Rooms without marks are alike. There are some marks already, probably made by your predecessors. Find the algorithm that saves you *for certain.*

An infinite object can be constructed inductively, i.e. by increasing and changing the object step by step. Usually there are an infinite number of incremental options each time, while the number of restrictions is finite at the same time.

2. A table consists of two rows and infinitely many columns. The numbers in each row are distinct, and the numbers in each column are distinct. Prove that one can select infinitely many columns and mark all the numbers in these columns, so that all the marked numbers be distinct.

*One can construct inductively by making a group at each step. One element of the group is **necessary** (for example, the minimal missing), while the others are optional.*

3. Oscar has constructed a sequence of 2018 distinct positive integers, such that for all $k \le 2018$ the sum of the first k terms is divisible by k. Prove that the sequence can be continued infinitely preserving the divisibility condition and having each positive integer occur exactly one time in the sequence.

4. Prove that every infinite sequence of real numbers contains an infinite monotonic subsequence.

Definition. An infinite set A is countable if it can be enumerated with all positive integers (i.e., there is a one to one correspondence between A and the set of all positive integers $\mathbb{N}$). Speaking informally, all elements of a countable set can be written as a sequence $a_1, a_2, \dots, a_n, \dots$

5. Are the following sets countable?

a) $\mathbb{Z}$; **b)** all points on the plane with integer coordinates;

c) $\mathbb{Q}$; **d)** $\mathbb{Z}[x]$ **e)** the set of all finite subsets of $\mathbb{N}$.

The Cantor diagonal method: at each step make the object under construction be distinct from the current term of the sequence.

6. Given a countable set of infinite binary sequences, prove that there is a binary sequence not in the set.

7. **a)** Two gods in turn erase positive integers, one integer per move. They do a countable amount of moves. The first god wins if at the end some prime be not erased. Can the second god avoid this?

b) Two gods in turn erase real numbers. During his n^th^ move a god chooses a segment of the length 2^{-n} on the number axis and erases all the numbers on the segment. They do a countable amount of moves. The first god wins if at the end some rational number be not erased. Can the second god avoid this?

c) Two gods in turn erase real numbers. During his n^th^ move a god chooses a segment of the length 2^{-n} on the number axis and erases all the numbers on the segment. They do a countable amount of moves. The first god wins if at the end some irrational number be not erased. Can the second god avoid this?

An inductive construction can be performed with any countable set, your should just rewrite it as a sequence.

8. **a)** Given the map $F: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, prove that there is an integer sequence $a_1, a_2, \dots, a_n, \dots$ such that $\forall k \in \mathbb{N}$ only finitely many inequalities $F(k, n) > a_n$ can be true.

b) Given the sequence $f_1, f_2, \dots, f_n, \dots$ of real continuous functions, defined for each $x>0$, prove that there is function $u(x)$ such that all solutions of inequality $f_k(x) > u(x)$ $\forall k$ belong to a finite segment.

c) An elementary function is a function, that one can get from 1, x, sin(x), e^x, arctan(x) by addition, subtraction, multiplication, division, composition and taking an inverse. Let us call a function $f(x)$ good if it is defined for every $x>0$ and is continuous. Prove that there is a good function $u(x)$ that increases faster then any good elementary function, i.e., for every good elementary function $g(x)$ we have $\lim_{x \to \infty} \frac{g(x)}{u(x)} = 0$

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Extra problems

IA1. Given a set M, there elements are some subsets of $\mathbb{N}$ so that $\forall A, B \in M$ either $A \subset B$ or $B \subset A$. Can M be an uncountable set?

IA2. The sequence is defined by the recurrence relation $x_{n+1}=[x_n]\{x_n\}$.

a) Prove that if $x_1>0$, then the sequence has a finite number of non-zeroes.

b) Prove that $|[x_{n+1}]| \le |[x_n]|$

c) Prove that if $-1<x_1<0$, then the sequence is periodic with the period 2

d) Solve the equation $x=[x]\{x\}$.

e) Prove that if $x_n<-1$, then there is a positive integer $m>n$ such that $[x_n] \ne [x_m]$

f) Prove that the sequence is periodic.

IMC2014.2.1 Let $d_n(x)$ denote n^{th} digit of the positive integer x, counting from right to left (for example, $d_3(2015)=0$. Suppose for the sequence $a_1, a_2, \dots, a_n, \dots$ there are only finitely many zeroes in the sequence $d_1(a_1), d_2(a_2), \dots, d_n(a_n), \dots$ Prove that there infinitely many positive integers that do not occur in the sequence $a_1, a_2, \dots, a_n, \dots$.

IMC2014.1.3 Let n be a positive integer. Show that there are positive real numbers $a_1, a_2, \dots, a_n$ such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1} x^{n-1} \pm \dots \pm a_1 x \pm a_0$$

has n distinct real roots.