Eigenvalues and Eigenvectors

1. Let A be an $n\times n$ square matrix and $\vec{v}\in\mathbb{R}^n, \vec{v}\neq 0.$ Prove that
a) if $A\vec{v}=\vec{0}$ then A is a degenerate matrix; **b)** If $A\vec{v}=l\vec{v}$ then det(A-l Id)=0.

Definition. Let A be a square matrix or a linear map. If $A\vec{v}=l\vec{v}, \vec{v}\neq 0$ then l is *eigenvalue* and $\vec{v}$ is *eigenvector* of A corresponding to l.

Usually A considered to be a complex matrix (even when all entries are real) with complex eigenvalues.

Theorem 2. All eigenvalues are roots of the *characteristic* polynomial $ch_A(x)=det(x\,Id-A)$.

Theorem 3. If A and B are similar then **a)** det(A)=det(B), **b)** tr(A)=tr(B); **c)** A and B have the same set of eigenvalues, **d)** the characteristic polynomials of A and B are the same.

4. Find en example of two nonsimilar matrices **a)** with same trace and determinant; **b)** with the same characteristic polynomial.

Theorem 5. (*without proof*) Let A be a $n\times n$ matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$. Then A is similar to an upper triangular matrix with the entries $\lambda_1,\ldots,\lambda_n$ on the main diagonal.

6. Let A be a $n\times n$ matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$. Prove that eigenvalues of A^2 are $\lambda_1{}^2,\ldots,\lambda_n{}^2$.

Theorem 7. (*without proof*) Let S be a real symmetric matrix. Then
a) S is similar to a real diagonal matrix,
b) There is an orthogonal real basis consisting of S's eigenvectors.

Theorem 8. (*without proof*) $ch_A(A)=0$.

9. Find the maximum $dim\langle Id, A, A^{2,} A^{3,}\ldots, A^{10}\rangle$ over all  matrices A.

10. Given a real 2×2 matrix and nonzero vectors $\vec{u},\vec{v}\in\mathbb{R}^2$ such that
$\vec{u}\perp\vec{v}, A\vec{u}=2\vec{u}, A\vec{v}=3\vec{v}$, prove that $|A\vec{w}|\geq 2|\vec{w}|$ $\forall\,\vec{w}\in\mathbb{R}^2$.

11. Determine all complex numbers l for which there exist a positive integer n and a real matrix A such that $A^2=A^T$ and l is an eigenvalue of A.

Credit problems

EV1. Does there exist a real 4×4 matrix A such that tr(A) = 5 and $A^2+A^T=Id$?

EV2. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let u be a real eigenvalue of matrix AB. Prove that $|u|>1$.

EV3. Prove that for any three 2×2 matrices A_1, A_2, A_3 , the polynomial $P(x_1,x_2,x_3)=\det(x_1A_1+x_2A_2+x_3A_3)$ is not identical to $x_1{}^2+x_2{}^2+x_3{}^2$.

EV4. Let A be a symmetric $n\times n$ matrix with real entries, let $a_1, a_2,\ldots,a_n$ denote values along its main diagonal, and let $l_1, l_2,\ldots,l_n$ denote its eigenvalues. Show that $\sum_{1\leq i<j\leq n} a_i a_j\geq\sum_{1\leq i<j\leq n} l_i l_j$ and determine all matrices for which the equality holds.

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