Matrix Algebra

Here all the matrices are $n\times n$ matrices unless otherwise stated.

Definition. A matrix A is *symmetric* if $A^T = A$, is *skew-symmetric* if $A^T = -A$, is *orthogonal* if $A^T A = Id$.

1. Find an example of a matrix that is **a)** symmetric, orthogonal and distinct from Id; **b)** skew-symmetric and orthogonal.

2. Prove that **a)** the set of all symmetric nxn-matrices is a vector space and find its dimension;
b) the set of all skew-symmetric nxn-matrices is a vector space and find its dimension;
c) the product of any two orthogonal matrices is an orthogonal matrix.

3. Prove that each square matrix has unique representation as the sum of symmetric and skew-symmetric matrices.

4. a) Find a real 2×2 matrix A such that $A^2 = -Id$;
b*) Prove that $\forall p, q \in \mathbb{R}$ there is a 2×2 matrix A such that $A^2 + pA + qId = 0$

5. a) For any positive integer m find a matrix A such that $A^{m-1} \neq 0, A^m = 0$
b*) For given n what is the minimum size of such A?

Definition. Square matrix A is *invertible* there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = Id$.

Theorem 6. A is invertible $\Leftrightarrow$ A is nondegenerate.

7. A is square matrix. **a)** $A^2 = 0$ **; b)** $A^m = 0, m \in \mathbb{N}$. Prove that $Id+A$ is invertible.

Definition. Let P be an invertible matrix. Denote $S^P = P^{-1} SP$

8. Prove that **a)** $(A+B)^P = A^P + B^P$; **b)** $(AB)^P = A^P B^P$; **c)** $(A^{-1})^P = (A^P)^{-1}$; d) $A^{PQ} = (A^P)^Q$.

9. Let Q be an orhtogonal matrix. Prove that **a)** $(A^T)^Q = (A^Q)^T$; **b)** If A be symmetric (skew-symmetric, orthogonal) so is A^Q.

Definition. A matrix A is *similar* to B if there is nonsingular matrix P such that $B = A^P$.

10. Find all matrices similar to Id.

11. Prove that **a)** If A is similar to B then B is similar to A; b) If A is similar to B and B is similar to C then A is similar to C.

Theorem 12. If A and B are similar then **a)** det(A)=det(B), **b)** tr(A)=tr(B).

13. Find en example of two nonsimilar matrices with same trace and determinant.

Theorem 14. (*without proof*) Let S be a real symmetric matrix. Then S is similar to a real diagonal matrix.

15. Does there exist a real 3×3 matrix A such that tr(A) = 0 and $A^2 + A = Id$?

Credit problems

MA1. Let n be an odd integer and S be a symmetric nxn matrix over $\mathbb{Z}_2$ with zeroes on the main diagonal. Prove that S is singular.

MA2. Prove that any nondegenerate matrix can be represented as a product of symmetric and orthogonal matrices.

MA3. Let A, B and C be real square matrices of the same size, and suppose that A is invertible. Prove that if $(A-B)C = BA^{-1}$, then $C(A-B) = A^{-1}B$.

MA4. a) For any integer $n>2$ and two $n\times n$ matrices with real entries A; B that satisfy the equation $(A+B)^{-1} = A^{-1} + B^{-1}$ prove that det(A) = det(B).
b) Does the same conclusion follow for matrices with complex entries?

MA5. Let k and n be positive integers. A sequence (A_1, …, A_k) of $n\times n$ real matrices is *good* if $A_i^2 \neq 0 \quad \forall i = 1,\dots,k$ but $A_i A_j = 0$ for $1 \leq i, j \leq k, i \neq j$. Show that $k \leq n$ in all good sequences, and give an example of a good sequence with $k = n$ for each n.