Rank and Determinant

Rank

1. Given a square matrix such as its columns be linearly independent, prove that its rows are linearly independent.

Definition. Such a matrix called *nondegenerate* or *nonsingular.*

Theorem 2. In any rectangular matrix A, the dimension of the linear span of rows is equal to the dimension of the linear span of columns.

Definition. Such dimension is called *rank* of A and is commonly denoted rank(A) or rk(A).

3. Prove that $|\text{rank } A - \text{rank } B| \le \text{rank}(A+B) \le \text{rank } A + \text{rank } B$.

Determinant is the function $\det : M_{n,n} \to K$

Good to remember

a) If M is a triangular matrix, det(M) is the product of diagonal entries;

b) Determinant is a linear function on each row and each column;

c) Determinant is skew-symmetric on rows and on columns;

d) A is nondegenerate $\Leftrightarrow \det A \neq 0$

e) $\det A^T = \det A$;

f) det(AB)=det A det B.

4. Prove that

a) $\det A^{-1} = \dfrac{1}{\det A}$;

b) $\forall n \in \mathbb{Z} \quad \det A^n = (\det A)^n$;

c) If n be an odd positive integer and A be nxn matrix and $A^T = -A$, then det A = 0.

5. Find the minimum number of nonzero entries in a nondegenerate matrix of size $n \times n$.

Definition. A *minor* of a matrix A is the determinant of some smaller square matrix, cut down from A by removing one or more of its rows or columns. The *size* of a minor is the size of the smaller matrix.

Theorem 6. Rank of matrix A is the maximum size of nonsero minor of A.

7. Prove that $\text{rank } AB \le \min(\text{rank } A, \text{rank } B)$.

Credit problems

RD1. Let A and B be matrices of size $n \times n$, AB=0. Prove that $\min(\text{rank } A, \text{rank } B) \le \dfrac{n}{2}$

RD2. What is the maximum possible value of determinant of a 3x3-matrix whose entries are either 0 or 1 ?

RD3. . Let n be an odd positive integer and A be nxn-matrix with entries a_{ij}=2 for i=j, a_{ij}=1 for $i-j=\pm 2 (mod\ n)$ and a_{ij}=0 otherwise. Find det A.

RD4. In the linear space of all real $n \times n$ matrices, find the maximum possible dimension of a linear subspace V such that $\forall X, Y \in V \quad \text{trace}(XY)=0$

(The trace of a matrix is the sum of the diagonal entries.)

RD5. Let A be an $n \times n$ -matrix with integer entries and $b_1, \dots, b_k$ be integers satisfying $\det A = b_1 \cdot \dots \cdot b_k$. Prove that there exist $n \times n$ -matrices $B_1, \dots, B_k$ with integer entries such that $A = B_1 \cdot \dots \cdot B_k$ and $\det B_i = b_i \quad \forall i = 1, \dots, k$.