Dimension and Matrices

Subspaces

Definition. A subset U of vector space V is called a *subspace* if U is closed under addition of vectors and multiplication of vectors with numbers.

1. Which of the following subsets are subspaces of K^n

a) rows with zero sum of entries, **b)** rows with the first entry equal to the last entry, **c)** rows with zero product of entries, **d)** rows of the form $(P(1), P(2), \ldots, P(n))$, where $P \in K_3[x]$ **.**

Lemma 2. The dimension of a subspace is less or equal then the dimension of the space.

3. Find the dimension of each subspace mentioned in Problem 1.

Definition. The *linear span* $K\langle \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \rangle$ is the set of all the vectors of the form $k_1 \vec{v}_1 + \ldots + k_m \vec{v}_m$, where $k_{1,} \ldots, k_m \in K$.

Lemma 4. The linear span $K\langle \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \rangle$ is a vector space of dimension $d \le m$.

Lemma 5. If $k > \dim V$ then any k vectors in V are linearly dependent.

6. Prove that **a)** $\dim K\langle \vec{v}_1, \ldots, \vec{v}_m, \vec{u}_1, \ldots, \vec{u}_n \rangle \le \dim K\langle \vec{v}_1, \ldots, \vec{v}_m \rangle + \dim K\langle \vec{u}_1, \ldots, \vec{u}_n \rangle$,

b) $\dim K\langle \vec{v}_1 + \vec{u}_1, \ldots, \vec{v}_m + \vec{u}_m \rangle \le \dim K\langle \vec{v}_1, \ldots, \vec{v}_m \rangle + \dim K\langle \vec{u}_1, \ldots, \vec{u}_m \rangle$

Theorem 7. The dimension of the space of solutions to the system of k linearly independent homogeneous equations for n unknowns is equal to $n - k$.

8. Oscar has 30 bananas and a a two-pan balance scale without weights. He suspects that all bananas are of same weight. What minimum number of weighings he needs to prove or disprove the suspicion? **(**A weighing is a test: Oscar puts two groups of bananas on the two pans of the balance scale and sees if the groups are of the same weight or not)

Matrices

Definition. Matrix is a rectangular table of numbers. Let $M_{m,n}$ be the set of all $m \times n$ matrices with entries from the field *K*. Then we have entry-wise addition and multiplication of matrices by numbers.

9. Prove that $M_{m,n}$ is a vector space and find $\dim M_{m,n}$.

10. The set $L_{m,n} \subset M_{m,n}$ consists of the matrices, the sum of its elements in every row is equal to zero and the sum of its elements in every column is equal to zero. Prove that $L_{m,n}$ is a vector subspace and find its dimension.

Credit problems

DM1. Alex writes a row of 10 numbers. In one move Andrew can choose any two numbers with exactly one number in between (for example, the 5th and the 7th numbers) and increase or decrease both with the same number. Andrew wins if he manages with such moves to get a row where all entries are the same. Can Alex prevent such a win by writing an appropriate row?

DM2. Let A be the $n \times n$ matrix, whose $(i, j)^{th}$ entry is $i+j$ for all $i, j = 1, 2, \ldots, n$. Find rank(I).

DM3. Let $n \ge 2$ be an integer. What is **a)** the minimal and **b)** maximal possible rank of an $n \times n$ matrix whose n^2 entries are precisely the numbers $1, 2, \ldots, n^2$?

DM4. Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

DM5. Given n+1 non-empty subsets of the set {1, 2, …, n}, prove that one can mark each subset as either white or black or gray so that the union of the white subsets be equal to the union of the black subsets.