Dimension of Vector Spaces

1. It is known that the triples (1,2,3) and (0,0, 4) satisfy the system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 = 0 \end{cases}$$

Prove that triples **a)** (1, 2, 7); **b)** (1, 2, -1); **c)** (5, 10, 15); **d)** (2, 4, 24) also satisfy this system. **e)** Find all solutions of the system if not all the coefficients are zero.

Definition. Let a set *V* of elements called *vectors* and a field *K* whose elements are called *numbers* be such that we can add vectors and multiply them by numbers to form from *V* a group by addition with the following properties (here $k, k_1, k_2 \in K, \vec{v}, \vec{v}_1, \vec{v}_2 \in V$):

1) $(k_1 + k_2)\vec{v} = k_1\vec{v} + k_2\vec{v}$,
2) $k(\vec{v}_1 + \vec{v}_2) = k\vec{v}_1 + k\vec{v}_2$,
3) $1 \cdot \vec{v} = \vec{v}$,
4) $(k_1 k_2)\vec{v} = k_1(k_2\vec{v})$.

Then *V* is said to be a *vector space* over *K*.

2. Which of the following objects are vector spaces, and over what fields?
a) linear equations of the form $a_1x_1 + a_2x_2 + \dots + a_nx_n = l$, **b)** K^n – rows or columns consisting of *n* numbers, **c)** increasing functions $\mathbb{R} \to \mathbb{R}$, **d)** $K_n[x]$ – polynomials of degree $d \le n$, **e)** complex numbers, **f)** $\mathbb{Z}[x]$, **g)** vectors with all rational coordinates in 3D-space, **h)** real sequences converging to zero, **i)** arithmetic sequences, **j)** geometric sequences .

Remark. Using the notion of vector space K^n one can replace a linear system of *n* equations for *k* unknowns with a single linear vector equation $x_1\vec{v}_1 + \dots + x_k\vec{v}_k = \vec{l}$.

Definition. A vector $\vec{v}$ is said to be *linearly dependent* (*over K*) of (the set of) the vectors $\vec{v}_1, \vec{v}_1, \dots, \vec{v}_m$ if it can be represented in the form $k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_m\vec{v}_m$, where $k_1, k_2, \dots, k_m \in K$

A collection of vectors $\vec{v}_1, \vec{v}_1, \dots, \vec{v}_m$ is said to be *linearly independent* if it does not contain vectors linearly dependent on other vectors of the collection; otherwise it is called *linearly dependent*.

3. Prove that each polynomial of degree ≤*n* is linearly dependent of the polynomials $1, x, x(x-1), x(x-1)(x-2), \dots, x(x-1)(x-2)\dots(x-n+1)$.

Definition. A *basis* of a vector space *V* is a collection of linearly independent vectors in *V* such that any other vector of *V* is linearly dependent of the collection.

Examples of bases: a) 1 and i of $\mathbb{C}$ over $\mathbb{R}$, **b)** (1, 0, …, 0), (0, 1, …, 0), …, (0, 0, …, 1) of K^n, **c)** $1, x, x^{2,}\dots, x^n$ of $K_n[x]$.

Theorem 4. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ is a basis of *V*, then any vector in *V*
a) can be represented in the form $k_1v_1 + \dots + k_mv_m$, where $k_{1,}\dots, k_m \in K$
b) Such a representation is unique.

Theorem 5 (*without proof*). **a)** Each vector space has a basis.
b) Any two bases of a vector space have the same number of vectors.
c) Any linearly independent set of vectors is a subset of a basis.

Definition. The number of vectors in a basis of a given vector space *V* is called the *dimension* of *V* and denoted $\dim V$.

6. Find the dimensions of each vector space mentioned in Problem 2.

Remark. The notion of dimension helps us to use the pigeonhole principle for vector spaces: though a space usually consists of an infinite number of vectors, the number of vectors in a basis is finite.

Lemma 7. Any n+1 vectors in K^n are linearly dependent.

Theorem 8. If a linear system of n equations for n unknowns has a single unique solution for some set of constant terms, then the system has a single unique solution for any set of constant terms.

Theorem 9. If a vector space V over K has a basis of n elements, then V is isomorphic to K^n .

Credit problems

Dim1. Let V be the set of all sequences with real terms a_n , where $n\in\mathbb{N}$, such that $a_{n+3}=a_n+2a_{n+1}+a_{n+2}$. Prove that V is a vector space and find $dim V$.

Dim2. Prove that for any distinct $a_{0,}a_{1,}\ldots,a_n\in K$ the set $(x-a_1)(x-a_2)\ldots(x-a_n),(x-a_0)(x-a_2)\ldots(x-a_n),\ldots,(x-a_0)(x-a_1)\ldots(x-a_{n-1})$ is a basis in $K_n[x]$

Dim3. For a real k, denote by $M_n(k)$ the set of nxn matrices A satisfying $A^T=kA$. Prove that $M_n(k)$ is a vector space and find its dimension for each k.

Dim4. Given 10 matrices of size 9x9. Prove that some nontrivial linear combination of these matrices has zero determinant.