Go-between in Inequalities

1. Prove that $\sqrt[3]{1001} > \sqrt[4]{9999}$.

2. Prove that $\frac{1}{2+1}+\frac{1}{2^2+1}+\ldots+\frac{1}{2^n+1}<1$

The main idea. To prove the inequality $A<B$ one can choose a *go-between* P and prove that, for example, $A<P$ and $P\le B$. The trick is to choose an appropriate P.

3. 10 married couples are dancing. Each husband is taller then his wife. Prove that if they will exchange partners so that the tallest woman will dance with the tallest man, the next tallest woman with the next tallest man and so on, then still in each dancing pair the man will be taller then the woman.

Idea. Go-betweens can build a chain: $A<P_1<P_2<\ldots<P_n<B$. One can create a go-between by transforming A or B, or get a new go-between from an old one.

4. A convex polygon A is placed within another convex polygon B. Prove that the perimeter of A is less then the perimeter of B.

Rearrangement inequality

5. a) Let $a\le b$, $x\le y$. Prove that $ay+bx\le ax+by$.

b) Let $x_1 \ge x_2 \ge \ldots \ge x_n, y_1 \ge y_2 \ge \ldots \ge y_n$. Prove the inequality

$x_1y_1 + x_2y_2 + \ldots + x_ny_n \ge x_1a_1 + x_2a_2 + \ldots + x_na_n \ge x_1y_n + x_2y_{n-1} + \ldots + x_ny_1$

(here $a_1, a_2, \ldots, a_n$ is a permutation of $y_1, y_2, \ldots, y_n$)

6. Let $a_1\ge a_2\ge\ldots\ge a_n, b_1\ge b_2\ge\ldots\ge b_n$. Prove *Chebyshev's inequality*

$n(a_1b_1+a_2b_2+\ldots+a_nb_n)\ge(a_1+a_2+\ldots+a_n)(b_1+b_2+\ldots+b_n)$

Credit problems

GI1. On each of n plates there is an apple and an orange. The weight of the apple can differ (either up or down) no more then by 5 g from the weight of the orange on the same plate. Prove that if both apples and oranges will be enumerated in ascending order according their weights, the difference between the weights of the fruits with the same number still can differ (either up or down) no more then by 5 g.

GI2. A square of the size 1 is cut into rectangles. One chooses a shorter side in each rectangle (if a rectangle is a square, one chooses any side). Prove that the sum of the lengths of the chosen sides is not less then 1.

GI3. a) Max and Lessy have each a copy of the same table 5×5 filled with 25 distinct real numbers. Max finds the maximal number M_1 in his table and crosses out the row and the column containing M_1, then he find the maximal number M_2 among the rest and crosses out the row and the column containing M_2, and so on. Lessy do the same with her copy of the table, but each time she finds the least of the numbers L_1, L_2, L_3, and so on. Prove that

$M_1+M_2+M_3+M_4+M_5\ge L_1+L_2+L_3+L_4+L_5$

b) Can it happen that $M_1+M_2+M_3+M_4+M_5=L_1+L_2+L_3+L_4+L_5$?

GI4. Let $0<a<b$. Prove that

$$\int_a^b (x^2+1)e^{-x^2}dx\ge e^{-a^2}-e^{-b^2}$$

GI5. Given a few positive numbers with the sum 10 and the sum of their squares 20, prove that the sum of their cubes is greater then 40.

www.ashap.info/Uroki/eng/NYUAD18/index.html