Remainders and Modular Arithmetic

Lemma 1. Let *a, d* be integers, $d>0$. Then there exist unique integers *q* and *r,* such that $a = qd + r$ and $0 \le r < d$.

Definition. The number *q* is called the *quotient*, while *r* is called the *remainder modulo d*.

2. The difference of two integers is divisible by *d* **iff** these integers have the same remainder modulo *d*.

Definition. This can be written short as $m \equiv n (mod\, d)$ or just $m \equiv_d n$.

3. Prove that the remainder of any prime module 30 is either a prime or 1.

Theorem 4 *(modular arithmetic)*. Let the integer a_1 has the remainder r_1 mod *d ,* and the integer a_2 has the remainder r_2 mod *d* . Then integers a_1+a_2 , a_1-a_2 , a_1a_2 has the same remainders modulo *d* as integers r_1+r_2, r_1-r_2, r_1r_2.respectively.

5. Find the last digit in the decimal expression of $777^{555^{333}}$.

6. Let *x* and *y* be positive integers. Prove that

a) $x^3 \equiv_{10} y^3 \Leftrightarrow x \equiv_{10} y$

b) If $10 | (x^2+xy+y^2)$ then $100 | (x^2+xy+y^2)$

7. The sum of a few odd perfect squares is 1011. Find the minimum number of terms in the sum.

If the polynomial equation has an integer solution, it has a solution modulo any positive integer. And vice versa: if the equation has no solution module some positive integer it has no integer solution.

8. Prove that

a) the equation $n^2+1=0$ (mod 3) has no solution.

b) he equation $n^2+1=3m$ has no integer solution.

9 Prove that the following equations has no integer solutions :

a) $x^2+y^2=4z-1$ **b)** $15x^2-7y^2=9$ **c)** $x^2+y^2+z^2=8t-1$

10. Given 100 integers such that the sum of any 99 of them is divisible by 13. Prove that each of the integers is divisible by 13.

Extra problems

Re1. A positive integer *N* is given to Huda. She divides *N* by 101 and gets the remainder $m>0$. Then Huda divides *N* by *m* and gets the remainder *p*. Find the maximum possible value for *p* and the least possible value for *N* to get this value for *p*.

Re2. Let *x*, *y*, *z* be integers such that $S = x^4 + y^4 + z^4$ is divisible by 29. Show that *S* is divisible by 29^4

Re3. Find the number of positive integers *n* satisfying the following conditions: $n<10^{2018}$ and $10^{2018}|n^2-n$.

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