Inequalities and second derivatives

Theorems

A. If $f''(x)>0 \quad \forall x\in(a,b)$ then $f(x)$ is convex on $[a,b]$, in particular, $f(pa+qb)<pf(b)+qf(b) \quad \forall p,q\in[0,1]$ such that $p+q=1$.

B. (*Jensen's inequality*) Let $f(x)$ be convex and numbers $a_1, a_2, \dots, a_n$ postive with the sum 1. Then $f(a_1x_1+\dots+a_nx_n)\le a_1f(x_1)+\dots+a_nf(x_n)$.

1. Prove that
   a) $\cos\frac{x+y}{2}\ge\frac{\cos x+\cos y}{2} \quad \forall x,y\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$
   b) $\sqrt{\frac{x+y+z}{3}}\ge\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3} \quad \forall x,y,z>0$
   c) $e^{0.3x+0.7y}\le 0.3e^x+0.7e^y \quad \forall x,y$
   d) $\tan 50^\circ+\tan 51^\circ+\tan 52^\circ+\dots+\tan 70^\circ>21\sqrt{3}$
2. Prove that **a)** $\ln\frac{x_1+x_2+\dots+x_n}{n}\ge\frac{\ln x_1+\ln x_2+\dots\ln x_n}{n}$
   b) $\frac{a_1+a_2+\dots+a_n}{n}\ge\sqrt[n]{a_1a_2\dots a_n}$
3. $a^ab^bc^cd^d\ge\left(\frac{a+b+c+d}{4}\right)^{a+b+c+d}$ (*a*, *b*, *c*, *d* are all positive)

Credit problems

SD1. Given positive numbers *p* and *q* such that $\frac{1}{p}+\frac{1}{q}=1$, prove that $ab\le\frac{a^p}{p}+\frac{b^q}{q}$ for all positive numbers *a*, *b*.

SD2. Polynomials *P* and *Q* are quadratic, and inequalities $P(x)\le 0$ and $Q(x)\le 0$ has no common solutions. Prove that there exist positive numbers *a* and *b* such that the polynomial $aP+bQ$ is positive for all *x*.

SD3. Let all $a_i>0$, $s = a_1+a_2+\dots+a_n$. Prove that $\frac{a_1}{s-a_1}+\frac{a_2}{s-a_2}+\dots+\frac{a_n}{s-a_n}\ge\frac{n}{n-1}$

SD4. Let *a*; *b*; *c*; *d*; *e* > 0 be real numbers such that $a^2+b^2+c^2=d^2+e^2$ and $a^4+b^4+c^4=d^4+e^4$. Compare the numbers $a^3+b^3+c^3$ and d^3+e^3.

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