Inequalities and derivatives

Theorems

A. If $f'(x)>0 \quad \forall x\in(a,b)$ then $f(x)$ is increasing on $[a,b]$,
in particular, $f(a)<f(b)$.
B. f(b)=f(a)+f'(z)(b-a) for some z between a and b. If m≤f'(x)≤M for any z on [a, b], then f(a)+m(b-a)≤f(b)≤f(a)+M(b-a).

1. Prove that $\sin x< x \quad \forall x> 0$
2. Prove that $|x+\frac{1}{x}|\geq 2 \quad \forall x\neq 0$
3. Prove that a) $e^x\geq x+1 \quad \forall x$

   b) $e^x>\frac{x^2}{2}+x+1 \quad \forall x>0$ and $e^x<\frac{x^2}{2}+x+1 \quad \forall x<0$

   c) $e^x\geq\frac{x^n}{n!}+...+\frac{x^2}{2}+x+1 \quad \forall x$ and any odd n

4. Compare e^{π} and π^e ?
5. Prove that $\cos\sqrt{x}\geq 1-\frac{x}{2} \quad \forall x\geq 0$

Credit problems

ID1. Prove that $\sin 2x+\cos x> 1 \quad \forall x\in(0,\frac{\pi}{3})$

ID2. Prove or disprove each of following statements
a) If *f* is continuous and $range(f)=\mathbb{R}$ then *f* is monotonic.
b) If *f* is monotonic and $range(f)=\mathbb{R}$ then *f* is continuous.
c) If *f* is monotonic and *f* is continuous then $range(f)=\mathbb{R}$.

ID3. Does $f(r)\leq g(r) \quad \forall r\in\mathbb{Q}$ imply $f(x)\leq g(x) \quad \forall x\in\mathbb{R}$ if
a) *f* and *g* are non-decreasing?
b) *f* and *g* are continuous?

ID4. Let $f:\mathbb{R}\to\mathbb{R}$ be a 2 times differentiable function satisfying $f(0)=1, f'(0)=0$ and
$f''(x)-5f'(x)+6f(x)\geq 0 \quad \forall x\geq 0$
Prove that
$f(x)\geq 3e^{2x}-2e^{3x} \quad \forall x\geq 0$

ID5. Compare functions $\sin(\tan x)$ and $\tan(\sin x)$ for all $x\in(0,\frac{\pi}{2})$

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