Polynomials with integer coefficients

0. $P(x)=x^2+x+41$. Is it true that P(x) is prime for any integer x?
1. Let *P* be a polynomials with integer coefficients (in short, $P\in\mathbb{Z}[x]$). Prove that $(a-b)|(P(a)-P(b))\quad\forall a,b\in\mathbb{Z}$
2. Let $P\in\mathbb{Z}[x]$, *P*(2)=1, n is positive integer, P(*n*)=0. Find *n* **.**
3. Let $P\in\mathbb{Z}[x]$ Prove that if *a* and *b* are integers of the same parity (i.e. are both odd or both even) then *P*(*a*) and *P*(*b*) are of the same parity.
4. Let $P\in\mathbb{Z}[x]$ **,** 6|*P*(2), 6|*P*(3). Prove that 6|*P*(5).
5. A polynomial *P* is such that *P*(7)=11 and *P*(11)=13. Prove that at least one of the coefficients of *P* is not integer.
6. Does there exist $P\in\mathbb{Z}[x], P\neq const$ such that $P(x)$ is a prime number for any positive integer *x*?
7. Let $H(x)=a_n x^n+\ldots+a_1 x+a_0$ be the polynomial with integer coefficients and $\frac{p}{q}$ an irreducible fraction such that $H(\frac{p}{q})=0$. Prove that **a)** $p|a_0$ and **b)** $q|a_n$
Definition. Fix a set K of coefficients, e.g., $K=\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{Z}_p,\mathbb{C}$. Let K[x] be the set of polynomials with coefficients in K. A polynomial of K[x] is said to be *irreducible* over K if it does not split into the product of lesser degree polynomials of K[x].
8. Over $\mathbb{Q}$, factorize the polynomials a) x^4+64 **b)** x^4+1 **c)** $x^6-11x^4+36x^2$ into irreducible polynomials.

Extra problems

PIC1. On the graph of a polynomial with integer coefficients two points with integer coordinates are marked. Prove that if the distance between them is integer, then the segment connecting them is parallel to the *x*-axis.
PIC2. $P\in\mathbb{Z}[x]$ **,** m/n is irreducible fraction, P(m/n)=0. Prove that there exists $Q\in\mathbb{Z}[x]$ such that P(x)=(nx-m)Q(x) .
PIC3. Let *P* be a nonconstant polynomial with integer coefficients. Prove that among numbers $P(1),P(2),P(3),\ldots$ there are infinitely many nonprime numbers.
PIC4. $P\in\mathbb{Z}[x]$ is irreducible over $\mathbb{Z}$. Prove that there exist $Q\in\mathbb{Z}[x]$ such that *P*(*Q*) is reducible over $\mathbb{Z}$.
PIC5. Let *p* be a polynomial with integer coefficients and let $a_1<a_2<\ldots<a_k$ be integers.
a) Prove that there exists $a\in\mathbb{Z}$ such that $p(a_i)$ divides *p*(*a*) for all *i*=1,2,...,*k*.
b) Does there exist $a\in\mathbb{Z}$ such that the product $p(a_1)\cdot p(a_2)\cdot\ldots\cdot p(a_k)$ divides *p*(*a*)?

www.ashap.info/Uroki/eng/NYUAD18/index.html