Sequences and sums

The sequence $a_1, a_2, \ldots, a_n, \ldots$ (or, just $\{a_n\}$) is defined if, for any positive integer n, one can define the number a_n. Usually, one defines a sequence either by an *explicit formula* $a_n=f(n)$, or by *a recurrence relation* for a general term and some given initial values.

Examples:

$a_{n+1}=a_n+d$ for $n \geq 1$ – *arithmetic progression*, or *arithmetic sequence* (d is its *common difference*)

$b_{n+1}=qb_n$ for $n \geq 1$ – *geometric progression*, or *geometric sequence* (q is its *common ratio*)

$f_1=f_2=1, f_{n+2}=f_n+f_{n+1}$ – for $n \geq 1$ – *Fibonacci numbers*

1. Find the explicit formula for arithmetic and geometric sequences provided their first terms are given.

Definition. For the sequence $\{a_n\}$ let S_n denote the sum $a_1+a_2+\ldots+a_n$.
If there exists $S=\lim S_n$ for $n \to \infty$, then S is called *the sum of the series* $a_1+a_2+\ldots$

2. Let $\{a_n\}$ be an arithmetic sequence. For the sequence $\{S_n\}$
a) find the recurrence relation; **b)** find the explicit formula.

3. Let $\{a_n\}$ be a geometric sequence. For the sequence $\{S_n\}$
a) find the recurrence relation; **b)** find the explicit formula.

4. a) Find the sum $1\frac{1}{2}+2\frac{1}{4}+3\frac{1}{8}+\ldots+100\frac{1}{2^{100}}$

b) Given are the first 6 terms of a sequence $\{a_n\}$:
9.9, 99.8, 999.7, 9999.6, 99999.5, 999999.4 .
Guess an explicit formula for a_n and find S_n
c*) Find the sum 1+11+111+...+11..11 (the last number consist of n digits).

Telescoping series

5. Find the sums **a)** $\ln\left(\frac{1}{3}\right)+\ln\left(\frac{3}{5}\right)+\ldots+\ln\left(\frac{99}{101}\right)$ **b)** $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{997\cdot 1000}$

c*) $\frac{0!}{3!}+\frac{1!}{4!}+\frac{2!}{5!}+\ldots+\frac{n!}{(n+3)!}$.

6. a) It is known that the sum of the first n terms in the sequence $\{a_n\}$ has the formula $S_n=n^2$.
Find the explicit formula for a_k.
b) The same question for $S_n=n^3$.
c) The same question for $S_n=n^3+2n^2$.

7.a) Find the explicit formula for $1^2+2^2+\ldots+n^2$
b) Find the explicit formula for $1^3+2^3+\ldots+n^3$
c*) Given a positive integer k, find the leading term in the explicit formula for $1^k+2^k+\ldots+n^k$

Binet's formula

8. Let F be the set of all sequences satisfying Fibonacci recurrence relation.
a) Find the common denominator for any geometric sequences from F.
b) Show that a linear combination of any two sequences from F belong to F (this means "F is a vector space").
c) Show that each sequence from F can be represented as a linear combination of geometric sequences from F (furthermore, geometric sequences build a basis in F).

d) Find the explicit formula for Fibonacci numbers.

More problems

SS1. Given a sequence $\{a_n\}$. It is known that there is an explicit formula for a_n and $a_4 - a_3 = a_3 - a_2 = a_2 - a_1$. Does this imply $a_n - a_{n-1} = a_2 - a_1$ for any positive integer n?

SS2. Given a finite arithmetic sequence. The sum of all terms is a power of two. Prove that the number of terms is also the power of two.

SS3. a) In the infinite geometric sequence, the first 9 terms are distinct positive integers. Does it mean that all the terms are integers?
b) In the infinite geometric sequence, the first 10 terms are distinct positive integers. Does it mean that all the terms are integers?

SS4. Does there exist a positive integer N and N–1 infinite arithmetic sequences with common differences 2, 3, 4, ..., N respectively such that each positive integer belong to at least one of these sequences?

SS5. In an infinite geometric sequence $b_1, b_2, \ldots, b_n, \ldots$. each term has been replaced with its fractional part. Can the new sequence be a descending geometric sequence?

SS6 (IMC10.1.2). Compute the sum of the series

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)}$$

SS7 (IMC14.1.2). Consider the following sequence

$$(a_n)_{n=1}^{\infty} = (1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,1,\ldots).$$

Find all pairs (u, v) of positive real numbers such that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{n^u} = v.$$

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