Example+estimate

The problems of this kind consist of two parts to be treated separately. First, one should give an example with the optimal value. Second, one should prove that a better example does not exist (this can be usually done by giving – proving – an upper or a lower estimate for all possible examples).

1. 11 straight lines divide a plane into parts. Each line is parallel either to x-axis or y-axis. Find the maximum number of parts.
2. An electronic clock shows time by displaying four digits, for example,13:10. Every minute Bob calculate the product of digits on the clock. Find the value most close to 77.
3. A polynomial of degree 17 with rational coefficients has no rational roots. It is factorized into polynomials of positive degrees, also with rational coefficients. Find the maximum number of factors.
4. Given a set of positive integers, they are pairwise relatively prime (a scientific term is "coprime") and their sum is equal to 100. Find the maximum amount integers in the set.

The problems below are of the same kind. But here one must produce a set of examples (or an algorithm how to get the optimal result in any case). Vice versa, the estimate here is proved by producing a "bad example", for which "a better then optimal value" can not be achieved.

5. Given are 111 balls, some of them red, some blue, some green and some white. It is known that if one blindly selects any 100 balls, then all 4 colors among the selected balls will be present for sure. Find the least N such that among any N blindly selected balls there will be at least 3 different colors.
6. a) Paul has 8 white cubes of size 1×1×1. He wants to put them together to build a big cube of size 2×2×2 that is white outside. Bill tries to prevent that. Bill may paint some of cubes sides black. What minimal amount of sides should he paint to be sure he has disabled Paul from constructing a big white cube?
b) The same question when Paul has 27 white cubes of size 1×1×1 and wants to construct a big white cube of size 3×3×3.

Extra problems

EE1. M is an $n\times n$ matrix, $\det(M)=0$. Find the maximal possible rank of M.
EE2. A function f has 10 roots, its derivative f' is continuous and has no common roots with f. Find the minimum number of local maximums f can have?
EE3. Let f be a real function, $f(1)=5$, $f'(x)\leq 2$ for any x. Find the minimal possible value of $f(-5)$.
EE4. A bank has 2002 employees. They all sit around a table. It is known that salaries of all employees are different and salaries of any two neighbors differ either with 2 or with 3 dollars. Find the maximal possible difference between the salaries of two employees.
EE5. Find the minimal amount of 1×1 squares one should draw to get a drawing of a checkered board of size 25×25 divided into 1×1 squares.
EE6. Given are several containers with goods. Each container weighs not more then 1000 kg, and the total weight of containers is 10000 kg. One lorry can transport at most 3000 kg. What is the minimal amount of such lorries that can for sure transport the whole lot of goods?
EE7. Any positive integer $N\geq 10$ in its decimal expression can be transformed into an exponent by moving the last digit up (e.g., the number 179 can be transformed into 17^9, or 2007 transformed into 200^7). Find the maximal N for which the transformed number is divisible by N.
EE8. Given the parabola $y=x^2$ and a circle. The circle is totally covered by the internal part of the parabola and is tangent to the parabola at the point (0, 0). Find the maximum radius of the circle.