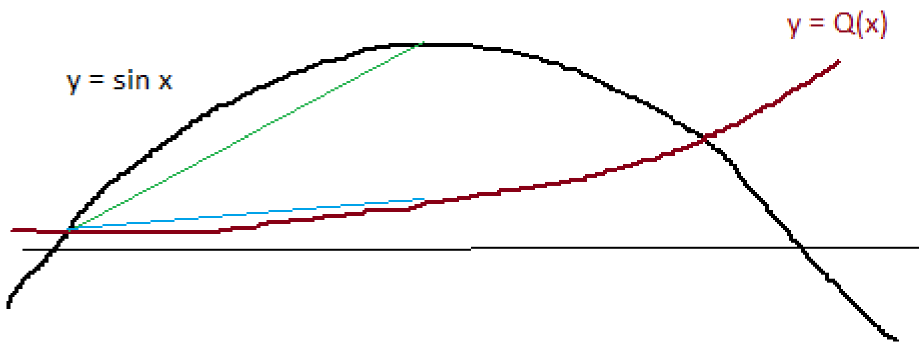
What is possible and what is impossible for a quadratic polynomial?
(some hints, answers and solutions)

Let $Q(x)=ax^2+bx+c$, where $a\neq 0$, be a quadratic with real coefficients.

1. **a)** Can the sum of coefficients of $Q(x)$ be positive while some of its values are negative?
   b) Can all the values of $Q(x)$ be positive if the sum of coefficients of $Q(x)$ is negative?
   Answer. a) Yes, see $Q=-x^2+2$
   b) No. Q(1) = a+b+c is negative.
2. For any integer x the value $Q(x)$ is an integer.
   a) Can all the coefficients of $Q(x)$ be noninteger?
   b) Can some of the coefficients of $Q(x)$ be noninteger?
   Answer. a) No. Q(0)=c
   b) Yes, see $Q(x)=\frac{x(x+1)}{2}$ For any integer x the number x(x+1) is even.

3. Can the whole graph of the equation $y=Q(x)$ lie higher than the graph of the equation $y=x^3$?
   Answer. No. Otherwise the equation $Q(x)=x^3$ would have no roots. But it has, as the polynomial $Q(x)-x^3$ is of an odd degree 3.
4. Given are 2 reduced quadratic polynomials. One of them has two distinct roots greater than 1000, the other one has two distinct roots smaller than 1000. Can the sum of these polynomials have one root greater than 1000 and the other root less than 1000? (A quadratic Q is *reduced* if $a=1$).
   Answer. No. Let P and Q were the polynomials, T=P+Q. The conditions imply P(1000)>0 and Q(1000)>0. Then T(1000)>0. But if 1000 lies between roots of T, it would be T(1000)<0.
5. Can the equation $Q(x)=\sin x$
   a) have exactly 1000 roots?
   b) have infinitely many roots?

   a) Answer. Yes. **Solution.**



Let $Q=\left(\frac{x+0.5\pi}{500\pi}\right)^2$. Then $Q(0.5\pi)<1, Q(2.5\pi)<1, \ldots, Q(498.5\pi)<1$ but $Q(500.5\pi)>1$. That means the graph y=Q(x) intersects exactly 250 “humps” of the graph y=sin x for x>0. Each hump is intersected exactly 2 times. It is obvious for the right part of a hump (sin x is descending, Q(x) is ascending). And the left part of a hump is concave while the graph y=Q(x) is convex. (See pic. The graph y=sinx lies *over* the green line segment, the graph y=sinx lies *under* the blue line segment, and the green segment is over the blue one). So to the right of $x=-\frac{\pi}{2}$ vi there are exactly 500 intersections. The same to the left of $x=-\frac{\pi}{2}$ as both graphs are symmetric.

b) Hint. The graph y=sin x can be divided into equal parts of the width 2π – “periods”. It is easy to show that the graph y=Q(x) can intersect only on a finite number of periods. But it is *not a complete solution.* We need to show that a number of intersections on a single period can not be infinite.

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