Polynomials and Equations

Some answers and solutions

Facts and theorems

Theorem I (Bézout). Let a be a root of the polynomial $P(x)$. Then there exists a polynomial $Q(x)$ such that $P(x)=(x-a)Q(x)$.

$$x^2-a^2=(x-a)(x+a)$$

$$x^3-a^3=(x-a)(x^2+ax+a^2)$$

$$x^n-a^n=(x-a)(x^{n-1}+ax^{n-2}+...+a^{n-2}x+a^{n-1})=(x-a)Q_n$$

$$P(x)=P(x)-P(0)=P(x)-P(a)=b_n(x^n-a^n)+b_{n-1}(x^{n-1}-a^{n-1})+...+b_1(x-a)=(x-a)(b_nQ_n+b_{n-1}Q_{n-1}+...)$$

Theorem II. a) Any polynomial of degree n has at most n roots.
b) If two polynomials of degree $\leq n$ coincide in $n+1$ points they are identical.

Proofs. a) It suffices to show that if a polynomial P has at least n roots has degree then $deg\,P\geq n$. Let a and b be distinct roots of $P(x)$. Then P(x)=(x-a)Q(x), and 0=P(b)=(b–a)Q(b)=0 => Q(b)=0 => Q(x)=(x-b)R(x) => P(x)=(x-a)Q(x) = (x-a)(x-b)R(x).

In the same way if $a_{1,}a_{2,}...,a_n$ be distinct roots of $P(x)$, then $P(x)=(x-a_1)(x-a_2)...(x-a_n)T(x)$ => $deg\,P\geq n+deg\,T\geq n$

Theorem IV (Viète). a) Numbers a and b are two roots of $x^2+px+q=0 \Leftrightarrow p=-(a+b),\ q=ab$.
b) Numbers a, b and c are three roots of $x^3+px^2+qx+r=0 \Leftrightarrow p=-(a+b+c),\ q=ab+ac+bc,\ r=-abc$.

Proof. b) $(x-a)(x-b)(x-c)=\ x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$

Theorem V. Given are distinct numbers $x_1, x_2, ..., x_n$. Then for any numbers $y_1, y_2, ..., y_n$ there exist a unique polynomial $P(x)$ of degree less then n such that $P(x_1)=y_1, P(x_2)=y_2,..., P(x_n)=y_n$.

Proof. P is unique beause of theorem IIb. Here is the example of the polynomial

$$L_i=\frac{(x-x_1)(x-x_2)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_1)(x_i-x_2)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}$$

$$L_i(x_1)=L_i(x_2)=...=L_i(x_n)=0\ except\ for\ L_i(x_i)=1$$

Then $P=y_1L_1+y_2L_2+...+y_nL_n$

$$P(x_i)=y_1L_1(x_i)+y_2L_2(x_i)+...+y_iL_i(x_i)+...=y_1\cdot 0+y_2\cdot 0+...y_i\cdot 1+...=y_i$$

Problems

3. In the polynomial identity $(x^2-1)(x+...)=(x-1)(x+3)(x+...)$ two numbers were replaced by dots. Restore the numbers.

Answer. $(x^2-1)(x+3)=(x-1)(x+3)(x+1)$

4. Find the sum of all coefficients of the polynomial $(x-2)^{100}$**.**

Hint. The sum of all coefficients of the polynomial P is equal to P(1)

Answer. $1=P(1)=(1-2)^{100}$

5. Given are two polynomials $P(x)$ and $Q(x)$ with real coefficients. Their sets of roots are A and B, respectively. Construct a polynomial with the set of roots equal to **a)** $A\cup B$ **b)** $A\cap B$.

Answer. a) PQ; **b)** P^2+Q^2

6. How many roots has the equation $\frac{(x-2)(x-5)}{(9-2)(9-5)}+\frac{(x-9)(x-5)}{(2-9)(2-5)}+\frac{(x-9)(x-2)}{(5-9)(5-2)}=1$?

Hint. Numbers 2, 5 and 9 satisfy the equation.

Answer. Infinitely many.

7. It is known that 10 is a root of $x^2+px+2-5\sqrt{3}=0$. Find the other root of this equation.

Answer. $0.2-0.5\sqrt{3}$

8. Factorize the polynomials: **a)** x^3+x^2+x-3 **b)** x^4+x^2+1 **c)** x^4+1.

Answer. a) $(x-1)(x^2+2x+3)$ **b)** $x^4+x^2+1 = (x^4+2x^2+1)-x^2=(x^2+1)^2-x^2 = (x^2+x+1)(x^2-x+1)$
c) $x^4+1 = (x^4+2x^2+1)-2x^2=(x^2+1)^2-(\sqrt{2}x)^2 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$

9. a) The product of four consecutive integers is 7!. Find these integers. How many solutions has the problem?

b) How many real solutions has the equation $x(x+1)(x+2)(x+3) = 7!$

Solution. $x(x+1)(x+2)(x+3) = [x(x+3)][(x+1)(x+2)] = (x^2+3x)(x^2+3x+2)$

$t=x^2+3x$, the equation $t(t+2)=5040$ has 2 solutions $t_1=70$ and $t_2=-72$.

Then $x^2+3x=70$ has 2 solutions $x_1=7$ and $x_2= -10$.

$x^2+3x= -72$ has no real solutions.

10. Given is a polynomial P. It is known that the equation $P(x)=x$ has no roots. Prove that the equation $P(P(x))=x$ has no roots.
Proof. $P(x)-x=0$ has no roots => either $P(x)-x>0$ for any x or $P(x)-x<0$ for any x. Hence either $P(x)>x$ for any x or $P(x)<x$ for any x.
Case 1) $P(x)>x \Rightarrow P(P(x))>P(x) \Rightarrow P(P(x))>x$ for any x.
Case 2) $P(x)<x \Rightarrow P(P(x))<P(x) \Rightarrow P(P(x))<x$ for any x.