Functional and Differential Inequalities

Let $f:\mathbb{R}\to\mathbb{R}$

Continuous functions

1. Given the function $f:\mathbb{R}\to\mathbb{R}$ such that, for every real $a>1$, the function $f(x)+f(ax)$ is continuous, prove that f is continuous.

2. (*RNO1993*)**.** Given $f:\mathbb{R}_{pos}\to\mathbb{R}_{pos}$ such that $f(x^y)=f(x)^{f(y)}\quad \forall x,y\in\mathbb{R}_{pos}$,

a) find all such continuous functions, **b)** find all such functions.

Functional inequalities

3. Given $f(x)>100\quad \forall x,$ prove that $f(f(x))+f(x^2)>200\quad \forall x$.

4. Given f be monotonic and $f(x)>x^3\quad \forall x,$ prove that $f(f(x))>x^9\quad \forall x$.

5. Given f be continuous and the equation $f(x)\neq x\quad \forall x,$ prove that $f(f(x))\neq x\quad \forall x$.

6. (*RNO2014*). Given the function $f:\mathbb{R}\to\mathbb{R}$ such that $(f(x))^2\le f(y)\ \forall x>y$, prove that $0\le f(x)\le 1\quad \forall x\in\mathbb{R}$

Differential inequalities

7. Given f such that $f(0)=0$, f' be continuous and $f'(x)>\cos(x)\ \forall x>0$, prove that $f(x)>\sin(x)\quad \forall x>0$.

8. Given f such that $f(0)=1$, f' be continuous and $f'(x)>(x^2+1)e^{x^2}\quad \forall x>0$, prove that $f(x)>e^{x^2}\quad \forall x>0$.

9. Given f such that $f(0)=3$, f' be continuous and $f'(x)+2f(x)\ge 0\quad \forall x$, prove that $f(x)<3e^{-2x}\quad \forall x<0$.

Extra problems

IMC6.2.2. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for any real numbers $a<b$, the image $f([a,b])$ is a closed interval of length $b-a$.

IMC8.1.1. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)-f(y)$ is rational for all real x and y such that $x-y$ is rational.

IMC9.2.2. Given f be two times differentiable such that $f(0)=1$, $f'(0)=0$ and $f''(x)-5f'(x)+6f(x)\ge 0\quad \forall x\ge 0$, prove that $f(x)\ge 3e^{2x}-2e^{3x}\quad \forall x\ge 0$.

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