Continuous Functional Equations

Let $f, g: \mathbb{R} \to \mathbb{R}$

1. Find out which of the following statements are true:
a) if *f* is continuous and f(a)>0, then $\exists c \neq a: f(c) > 0$,
b) if *f* is continuous and f(a)=A, f(b)=B, then $\forall C \in [A, B] \exists c \in [a, b]: f(c) = C$,

c) if *f* is continuous and $\text{range}(f) = \mathbb{R}$ then *f* is monotonic,
d) if *f* is monotonic and $\text{range}(f) = \mathbb{R}$ then *f* is continuous,
e) if *f* is continuous and takes each positive value then *f* has a zero,
f) if *f* and *g* are continuous and $f(r) \leq g(r) \quad \forall r \in \mathbb{Q}$ then $f(x) \leq g(x) \quad \forall x \in \mathbb{R}$.
g) if *f* and *g* are non-decreasing and $f(r) \leq g(r) \quad \forall r \in \mathbb{Q}$ then $f(x) \leq g(x) \quad \forall x \in \mathbb{R}$
h) if $f(x) - f(y) \leq 10|x - y| \quad \forall x, y$ then *f* is continuous
i) if *f* is continuous and $\text{range}(f) \subset \mathbb{Q}$ then *f* is constant.

2. a) Find all continuous functions *f* such that $f(x+y) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}$.
b) Find all monotonic functions *f* such that $f(x+y) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}$.

3. Given a function *f* such that $f(x+2y) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}$, find
a) $f(-1)$ **b)** *f*(2015) **c)** *f*(*x*).

4. Find all continuous functions $F: M \to \mathbb{R}$ if $f(x+2y) = f(x) + f(y) \quad \forall x, y \in M$ and
a) $M = \{x \in \mathbb{R} : x \geq 0\}$ **b)** $M = \mathbb{R}_{pos}$.

5. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) f(y) \quad \forall x, y \in \mathbb{R}$.

6. Given the function $f: \mathbb{R} \to \mathbb{R}$ such that, for every real *a*>1, the function *f*(*x*)+*f*(*ax*) is continuous, prove that *f* is continuous.

Extra problems

RNO2014. Given the function $f: \mathbb{R} \to \mathbb{R}$ such that $(f(x))^2 \leq f(y) \ \forall x > y$, prove that $0 \leq f(x) \leq 1 \quad \forall x \in \mathbb{R}$

RNO1993. Given $f: \mathbb{R}_{pos} \to \mathbb{R}_{pos}$ such that $f(x^y) = f(x)^{f(y)} \quad \forall x, y \in \mathbb{R}_{pos}$, find

a) Find all such continuous functions, **b)** all such functions.

IMC6.2.2 Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that for any real numbers $a < b$, the image $f([a, b])$ is a closed interval of length $b - a$.

IMC8.1.1 Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y)$ is rational for all real *x* and *y* such that $x - y$ is rational.

RNO1998. Find every $a \in \mathbb{R}$ such that there exists a non-constant function $f: \mathbb{R} \to \mathbb{R}$ satisfying $f(a(x+y)) = f(x) + f(y)$.

IMC8.1.2 Denote by *V* the real vector space of all real polynomials in one variable, and let $P: V \to \mathbb{R}$ be a linear map. Suppose that for all $f, g \in V$ with $P(fg) = 0$, we have $P(f) = 0$ or $P(g) = 0$. Prove that there exist real numbers x_0, c such that $P(f) = cf(x_0)$ for all $f \in V$.