Vectors. Dot product

Definition. A (*Euclidean*) *vector* $\vec{a}$ is a directed line segment $\vec{KL}$, where K is the *initial point*, L is the *endpoint*. It has *length* $|\vec{a}|$ (the length of the segment KL) and *direction*. A vector of length 1 is called a *unit vector*. Two vectors with the same length and direction are considered to be equal, so a vector can be *translated* to begin from any point . We can *add two vectors* ($\vec{KL}+\vec{LM}=\vec{KM}$), subtract them ($\vec{KM}-\vec{KL}=\vec{LM}$ and *multiply a vector with a real number* ($k\vec{a}$ has length $|k|\cdot|\vec{a}|$, it has the same direction as $\vec{a}$ if $k>0$ and opposite direction if $k<0$). Usually we consider the set of vectors lying on the plane or in the space.

1. Prove the triangle inequality $|\vec{a}_1+\vec{a}_2+...+\vec{a}_n|\le|\vec{a}_1|+|\vec{a}_2|+...+|\vec{a}_n|$. When does the equality holds?

2. The sum of four unit plane vectors is $\vec{0}$. Prove that one can split vectors into two pairs with zero sum in each pair.

3. One can move the medians of a triangle without rotation. Prove that one can move them in such a way they'll be the sides of another triangle.

4. A regular polygon $A_1A_2...A_n$ is inscribed into a circle with the center O. Prove that $\vec{OA}_1+\vec{OA}_2+...+\vec{OA}_n=\vec{0}$.

5. Let $A_1A_2...A_7$ be a convex polygon. Prove that one can select a direction on each of its diagonals so that the sum of all corresponding vectors be zero.

6. For each side of a polygon, draw the vector orthogonal to the side and pointing outside the polygon with the length of the side. Prove that the sum of all the vectors drawn is zero.

Definition. The *dot product* $\vec{a}\cdot\vec{b}$ is the *number* $|\vec{a}||\vec{b}|\cos\varphi$, where φ is the angle between $\vec{a}$ and $\vec{b}$. In particular, one can write $\vec{a}^2$ instead of $\vec{a}\cdot\vec{a}$.

7. Prove the properties of the dot product:

a) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ **b)** $\vec{a}^2=|\vec{a}|^2$ **c)** the vectors $\vec{a}$ and $\vec{b}$ are orthogonal $\Leftrightarrow$ $\vec{a}\cdot\vec{b}=0$.

Theorem 8 (*Linearity of the dot product*).

Let *k, l* be real numbers; then $(k\vec{a}+l\vec{b})\cdot\vec{c}=k\vec{a}\cdot\vec{c}+l\vec{b}\cdot\vec{c}$.

9. Given $\vec{a}+\vec{b}+\vec{c}+\vec{d}=\vec{0}$, prove that $|\vec{d}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{a}\cdot\vec{c}+2\vec{b}\cdot\vec{c}$.

10. Given $|\vec{a}|=|\vec{b}|=l$ and the angle between $\vec{a}$ and $\vec{b}$ equal to φ, find $|\vec{a}+\vec{b}|$.

11. Prove that if the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are orthogonal, then $|\vec{a}|=|\vec{b}|$ **.**

12. Prove that if $\vec{b}\neq\vec{0}$, then the vector $\vec{a}-\frac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\vec{b}$ is orthogonal to $\vec{b}$.

13. Let *A,B,C,D* be four points on the plane. Prove that $\overline{AB}\cdot\overline{CD}+\overline{BC}\cdot\overline{AD}+\overline{CA}\cdot\overline{BD}=0$.

14. Let the triangle ABC be inscribed into a circle with the center O**,** and H the point such that $\vec{OH}=\vec{OA}+\vec{OB}+\vec{OC}$. Prove that H is the intersection point of the triangle's altitudes.

15. Given 3 vectors $\vec{a}_1,\vec{a}_2,\vec{a}_3$ in 3D-space which are not parallel to a plane, prove that there are 3 vectors $\vec{b}_1,\vec{b}_2,\vec{b}_3$ such that $\vec{a}_i\cdot\vec{b}_j=\delta_{ij}$ $\forall i,j$ **.** (Here $\delta_{ii}=1$ $\forall i$ and 0 otherwise)

Extra problems

VD1. Given 4 vectors $\vec{a},\vec{b},\vec{c},\vec{d}$ with zero sum, each of them not parallel to any other, prove that $|\vec{a}|+|\vec{b}|+|\vec{c}|+|\vec{d}|>|\vec{a}+\vec{b}|+|\vec{a}+\vec{c}|+|\vec{a}+\vec{d}|$.

VD2. Given quadrangles *ABCD* and *A'B'C'D'* such that *AB=A'B'*, *BC=B'C'*, *CD=C'D'*, *AD=A'D'* and $AC\perp BD$, prove that $A'C'\perp B'D'$.

VD3. Given five vectors on the plane, prove that one can select two of them so that the length of their sum is less or equal to the length of the sum of the three other vectors.

VD4. Let $\vec{a}_1,\vec{a}_2,...,\vec{a}_n$ be plane vectors with the length of each not greater then 1, prove that one can select signs in the sum $\vec{c}=\vec{a}_1\pm\vec{a}_2\pm...\pm\vec{a}_n$ so that $|\vec{c}|\le\sqrt{2}$.