Go-between in Inequalities

0. A square of the size 1 is cut into rectangles. One chooses a shorter side in each rectangle (if a rectangle is a square, one chooses any side). Prove that the sum of the lengths of the chosen sides is not less then 1.

The main idea. To prove the inequality $A<B$ one can choose a *go-between* P and prove that, for example, $A<P$ and $P\le B$. The art is to choose an appropriate P. In the problem 0 we can use the sum of the part's areas as the go-between.

1. Prove that $\sqrt[3]{1001} > \sqrt[4]{9999}$.

2. 10 married couples are dancing. Each husband is taller then his wife. Prove that if they will exchange partners so that the tallest woman will dance with the tallest man, the next tallest woman with the next tallest man and so on, then still in each dancing pair the man will be taller then the woman.

Idea. Go-betweens can build a chain: $A<P_1<P_2<\ldots<P_n<B$. One can create a go-between by transforming A or B, or get a new go-between from an old one.

3. A convex polygon A is placed within another convex polygon B. Prove that the perimeter of A is less then the perimeter of B.

Rearrangement inequality

4. a) Let $a\le b$, $x\le y$. Prove that $ay+bx\le ax+by$.

b) Let $x_1 \ge x_2 \ge \ldots \ge x_n, y_1 \ge y_2 \ge \ldots \ge y_n$. Prove the inequality

$$x_1y_1 + x_2y_2 + \ldots + x_ny_n \ge x_1a_1 + x_2a_2 + \ldots + x_na_n \ge x_1y_n + x_2y_{n-1} + \ldots + x_ny_1$$

(here $a_1, a_2, \ldots, a_n$ is a permutation of $y_1, y_2, \ldots, y_n$)

5. Let $a_1\ge a_2\ge\ldots\ge a_n, b_1\ge b_2\ge\ldots\ge b_n$. Prove *Chebyshev's inequality*

$$n(a_1b_1+a_2b_2+\ldots+a_nb_n)\ge(a_1+a_2+\ldots+a_n)(b_1+b_2+\ldots+b_n)$$

Sturm's method

6. a) Suppose we replace each factor in the product 100·101·102·...·200 with 150. Will the product increase or decrease? The same question for the sum 100+101+...+200.

б) Will the sum $\frac{1}{100}+\frac{1}{101}+\cdots+\frac{1}{199}+\frac{1}{200}$ increase or decrease, if each term be replaced with $\frac{1}{150}$?

7. Let the sum of two positive real numbers a and b is fixed. Which of the following expressions increase and which decrease if a and b be moved closer to each other?

a) ab **b)** a^2+b^2 **c)** $\frac{1}{a}+\frac{1}{b}$ **d)** a^4+b^4 **e)** $\sqrt{a}+\sqrt{b}$ **f)** a^n+b^n **g)** $\frac{1}{a^n}+\frac{1}{b^n}$.

Idea. In a sum go-betweens can be chosen only for one term or a group of terms.

8 Prove that for all 100-gons inscribed in the circle the regular 100-gon has the greatest area.

9. Prove the inequality for *arithmetic mean*, *geometric mean* and *harmonic mean*:

if $a_1, a_2, \ldots, a_n>0$ then $$\frac{a_1+a_2+\ldots+a_n}{n}\ge\sqrt[n]{a_1a_2\ldots a_n}\ge\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}$$

10. For all polygons inscribed in the circle find the polygon with the greatest ratio of the area to the number of sides.

Extra problems

IMC2011.1.1 Let 0<*a*<*b*. Prove that

$$\int_a^b (x^2+1)e^{-x^2}\,dx \geq e^{-a^2}-e^{-b^2}$$

GI1. On each of *n* plates there is an apple and an orange. The weight of the apple can differ (either up or down) no more then by 5 g from the weight of the orange on the same plate. Prove that if both apples and oranges will be enumerated in ascending order according their weights, the difference between the weights of the fruits with the same number still can differ (either up or down) no more then by 5 g.

GI2. a) Max and Lessy have two copies of the same table 5×5 filled with 25 distinct real numbers. Max finds the maximal number M_1 in the table and crosses out the row and the column containing M_1, then he find the maximal number M_2 among the rest and crosses out the row and the column containing M_2, and so on. Lessy do the same with your copy of the table, but each time she finds the least of the numbers L_1, L_2, L_3, and so on. Prove that

$$M_1+M_2+M_3+M_4+M_5 \geq L_1+L_2+L_3+L_4+L_5$$

b) Can it happen that $M_1+M_2+M_3+M_4+M_5=L_1+L_2+L_3+L_4+L_5$?

GI3. Given a few positive numbers with the sum 10 and the sum of their squares 20, prove that the sum of their cubes is greater then 40.

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