XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, July 31

Problems

First day. 8 grade

8.1. Let $ABCD$ be a cyclic quadrilateral with $AB = BC$ and $AD = CD$. A point M lies on the minor arc CD of its circumcircle. The lines BM and CD meet at point P, the lines AM and BD meet at point Q. Prove that $PQ \parallel AC$.

8.2. Let H and O be the orthocenter and the circumcenter of an acute-angled triangle ABC, respectively. The perpendicular bisector to segment BH meets AB and BC at points A_1 and C_1, respectively. Prove that OB bisects the angle A_1OC_1.

8.3. Let AD, BE and CF be the medians of triangle ABC. The points X and Y are the reflections of F about AD and BE, respectively. Prove that the circumcircles of triangles BEX and ADY are concentric.

8.4. Alex dissects a paper triangle into two triangles. Each minute after this he dissects one of obtained triangles into two triangles. After some time (at least one hour) it appeared that all obtained triangles were congruent. Find all initial triangles for which this is possible.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, July 31

Problems

First day. 8 grade

8.1. Let $ABCD$ be a cyclic quadrilateral with $AB = BC$ and $AD = CD$. A point M lies on the minor arc CD of its circumcircle. The lines BM and CD meet at point P, the lines AM and BD meet at point Q. Prove that $PQ \parallel AC$.

8.2. Let H and O be the orthocenter and the circumcenter of an acute-angled triangle ABC, respectively. The perpendicular bisector to segment BH meets AB and BC at points A_1 and C_1, respectively. Prove that OB bisects the angle A_1OC_1.

8.3. Let AD, BE and CF be the medians of triangle ABC. The points X and Y are the reflections of F about AD and BE, respectively. Prove that the circumcircles of triangles BEX and ADY are concentric.

8.4. Alex dissects a paper triangle into two triangles. Each minute after this he dissects one of obtained triangles into two triangles. After some time (at least one hour) it appeared that all obtained triangles were congruent. Find all initial triangles for which this is possible.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 8 grade

8.5. A square $ABCD$ is given. Two circles are inscribed into angles A and B, and the sum of their diameters is equal to the sidelength of the square. Prove that one of their common tangents passes through the midpoint of AB.

8.6. A median of an acute-angled triangle dissects it into two triangles. Prove that each of them can be covered by a semidisc congruent to a half of the circumdisc of the initial triangle.

8.7. Let $A_1A_2\dots A_{13}$ and $B_1B_2\dots B_{13}$ be two regular 13-gons in the plane such that the points B_1 and A_{13} coincide and lie on the segment A_1B_{13}, and both polygons lie in the same semiplane with respect to this segment. Prove that the lines A_1A_9, $B_{13}B_8$ and A_8B_9 are concurrent.

8.8. Let $ABCD$ be a square, and let P be a point on the minor arc CD of its circumcircle. The lines PA, PB meet the diagonals BD, AC at points K, L respectively. The points M, N are the projections of K, L respectively to CD, and Q is the common point of lines KN and ML. Prove that PQ bisects the segment AB.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 8 grade

8.5. A square $ABCD$ is given. Two circles are inscribed into angles A and B, and the sum of their diameters is equal to the sidelength of the square. Prove that one of their common tangents passes through the midpoint of AB.

8.6. A median of an acute-angled triangle dissects it into two triangles. Prove that each of them can be covered by a semidisc congruent to a half of the circumdisc of the initial triangle.

8.7. Let $A_1A_2\dots A_{13}$ and $B_1B_2\dots B_{13}$ be two regular 13-gons in the plane such that the points B_1 and A_{13} coincide and lie on the segment A_1B_{13}, and both polygons lie in the same semiplane with respect to this segment. Prove that the lines A_1A_9, $B_{13}B_8$ and A_8B_9 are concurrent.

8.8. Let $ABCD$ be a square, and let P be a point on the minor arc CD of its circumcircle. The lines PA, PB meet the diagonals BD, AC at points K, L respectively. The points M, N are the projections of K, L respectively to CD, and Q is the common point of lines KN and ML. Prove that PQ bisects the segment AB.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, July 31

Problems

First day. 9 grade

9.1. Let ABC be a regular triangle. The line passing through the midpoint of AB and parallel to AC meets the minor arc AB of its circumcircle at point K. Prove that the ratio $AK : BK$ is equal to the ratio of the side and the diagonal of a regular pentagon.

9.2. Let I be the incenter of triangle ABC, M be the midpoint of AC, and W be the midpoint of arc AB of its circumcircle not containing C. It is known that $\angle AIM = 90^\circ$. Find the ratio $CI : IW$.

9.3. The angles B and C of an acute-angled triangle ABC are greater than 60°. Points P and Q are chosen on the sides AB and AC, respectively, so that the points A, P, Q are concyclic with the orthocenter H of the triangle ABC. Point K is the midpoint of PQ. Prove that $\angle BKC > 90^\circ$.

9.4. Points M and K are chosen on lateral sides AB and AC, respectively, of an isosceles triangle ABC, and point D is chosen on its base BC so that $AMDK$ is a parallelogram. Let the lines MK and BC meet at point L, and let X, Y be the intersection points of AB, AC respectively with the perpendicular line from D to BC. Prove that the circle with center L and radius LD and the circumcircle of triangle AXY are tangent.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, July 31

Problems

First day. 9 grade

9.1. Let ABC be a regular triangle. The line passing through the midpoint of AB and parallel to AC meets the minor arc AB of its circumcircle at point K. Prove that the ratio $AK : BK$ is equal to the ratio of the side and the diagonal of a regular pentagon.

9.2. Let I be the incenter of triangle ABC, M be the midpoint of AC, and W be the midpoint of arc AB of its circumcircle not containing C. It is known that $\angle AIM = 90^\circ$. Find the ratio $CI : IW$.

9.3. The angles B and C of an acute-angled triangle ABC are greater than 60°. Points P and Q are chosen on the sides AB and AC, respectively, so that the points A, P, Q are concyclic with the orthocenter H of the triangle ABC. Point K is the midpoint of PQ. Prove that $\angle BKC > 90^\circ$.

9.4. Points M and K are chosen on lateral sides AB and AC, respectively, of an isosceles triangle ABC, and point D is chosen on its base BC so that $AMDK$ is a parallelogram. Let the lines MK and BC meet at point L, and let X, Y be the intersection points of AB, AC respectively with the perpendicular line from D to BC. Prove that the circle with center L and radius LD and the circumcircle of triangle AXY are tangent.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 9 grade

9.5. Let BH_b, CH_c be altitudes of an acute-angled triangle ABC. The line H_bH_c meets the circumcircle of ABC at points X and Y. Points P and Q are the reflections of X and Y about AB and AC, respectively. Prove that $PQ \parallel BC$.

9.6. Let ABC be a right-angled triangle ($\angle C = 90^\circ$) and D be the midpoint of an altitude from C. The reflections of the line AB about AD and BD, respectively, meet at point F. Find the ratio $S_{ABF} : S_{ABC}$.

9.7. Let a and b be parallel lines with 50 distinct points marked on a and 50 distinct points marked on b. Find the greatest possible number of acute-angled triangles all whose vertices are marked.

9.8. Let AK and BL be the altitudes of an acute-angled triangle ABC, and let ω be the excircle of ABC touching the side AB. The common internal tangents to circles CKL and ω meet AB at points P and Q. Prove that $AP = BQ$.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 9 grade

9.5. Let BH_b, CH_c be altitudes of an acute-angled triangle ABC. The line H_bH_c meets the circumcircle of ABC at points X and Y. Points P and Q are the reflections of X and Y about AB and AC, respectively. Prove that $PQ \parallel BC$.

9.6. Let ABC be a right-angled triangle ($\angle C = 90^\circ$) and D be the midpoint of an altitude from C. The reflections of the line AB about AD and BD, respectively, meet at point F. Find the ratio $S_{ABF} : S_{ABC}$.

9.7. Let a and b be parallel lines with 50 distinct points marked on a and 50 distinct points marked on b. Find the greatest possible number of acute-angled triangles all whose vertices are marked.

9.8. Let AK and BL be the altitudes of an acute-angled triangle ABC, and let ω be the excircle of ABC touching the side AB. The common internal tangents to circles CKL and ω meet AB at points P and Q. Prove that $AP = BQ$.

XIII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

First day. 10 grade

10.1. Let A and B be the common points of two circles, and CD be their common tangent (C and D are the tangency points). Let O_a, O_b be the circumcenters of triangles CAD, CBD respectively. Prove that the midpoint of segment O_aO_b lies on the line AB.

10.2. Prove that the distance from any vertex of an acute-angled triangle to the corresponding excenter is less than the sum of two greatest sidelengths.

10.3. Let $ABCD$ be a convex quadrilateral, and let $\omega_A, \omega_B, \omega_C, \omega_D$ be the circumcircles of triangles BCD, ACD, ABD, ABC, respectively. Denote by X_A the product of the power of A with respect to ω_A and the area of triangle BCD. Define X_B, X_C, X_D similarly. Prove that $X_A + X_B + X_C + X_D = 0$.

10.4. A scalene triangle ABC and its incircle ω are given. Using only a ruler and drawing at most eight lines, rays or segments, construct points A', B', C' on ω such that the rays $B'C'$, $C'A'$, $A'B'$ pass through A, B, C, respectively.

XIII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

First day. 10 grade

10.1. Let A and B be the common points of two circles, and CD be their common tangent (C and D are the tangency points). Let O_a, O_b be the circumcenters of triangles CAD, CBD respectively. Prove that the midpoint of segment O_aO_b lies on the line AB.

10.2. Prove that the distance from any vertex of an acute-angled triangle to the corresponding excenter is less than the sum of two greatest sidelengths.

10.3. Let $ABCD$ be a convex quadrilateral, and let $\omega_A, \omega_B, \omega_C, \omega_D$ be the circumcircles of triangles BCD, ACD, ABD, ABC, respectively. Denote by X_A the product of the power of A with respect to ω_A and the area of triangle BCD. Define X_B, X_C, X_D similarly. Prove that $X_A + X_B + X_C + X_D = 0$.

10.4. A scalene triangle ABC and its incircle ω are given. Using only a ruler and drawing at most eight lines, rays or segments, construct points A', B', C' on ω such that the rays $B'C'$, $C'A'$, $A'B'$ pass through A, B, C, respectively.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 10 grade

10.5. Let BB', CC' be the altitudes of an acute-angled triangle ABC. Two circles passing through A and C' are tangent to BC at points P and Q. Prove that A, B', P, Q are concyclic.

10.6. Let the insphere of a pyramid $SABC$ touch the faces SAB, SBC, SCA at points D, E, F respectively. Find all possible values of the sum of angles SDA, SEB and SFC.

10.7. A quadrilateral $ABCD$ is circumscribed around circle ω centered at I and inscribed into circle Γ. The lines AB and CD meet at point P, the lines BC and AD meet at point Q. Prove that the circles PIQ and Γ are orthogonal.

10.8. Suppose S is a set of points in the plane, $|S|$ is even; no three points of S are collinear. Prove that S can be partitioned into two sets S_1 and S_2 so that their convex hulls have equal number of vertices.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 10 grade

10.5. Let BB', CC' be the altitudes of an acute-angled triangle ABC. Two circles passing through A and C' are tangent to BC at points P and Q. Prove that A, B', P, Q are concyclic.

10.6. Let the insphere of a pyramid $SABC$ touch the faces SAB, SBC, SCA at points D, E, F respectively. Find all possible values of the sum of angles SDA, SEB and SFC.

10.7. A quadrilateral $ABCD$ is circumscribed around circle ω centered at I and inscribed into circle Γ. The lines AB and CD meet at point P, the lines BC and AD meet at point Q. Prove that the circles PIQ and Γ are orthogonal.

10.8. Suppose S is a set of points in the plane, $|S|$ is even; no three points of S are collinear. Prove that S can be partitioned into two sets S_1 and S_2 so that their convex hulls have equal number of vertices.