XII Geometrical Olympiad in honour of I.F.Sharygin
The correspondence round. Solutions

1. (A.Trigub, 8) A trapezoid $ABCD$ with bases AD and BC is such that $AB = BD$. Let M be the midpoint of DC. Prove that $\angle MBC = \angle BCA$.

   Solution. Let the line BM meet AD at point K. Then $BCKD$ is a parallelogram, therefore $CK = BD = AB$. Thus we obtain, since $ABCK$ is an equilateral trapezoid, that $\angle BCA = \angle CBK = \angle MBC$ (fig.1).

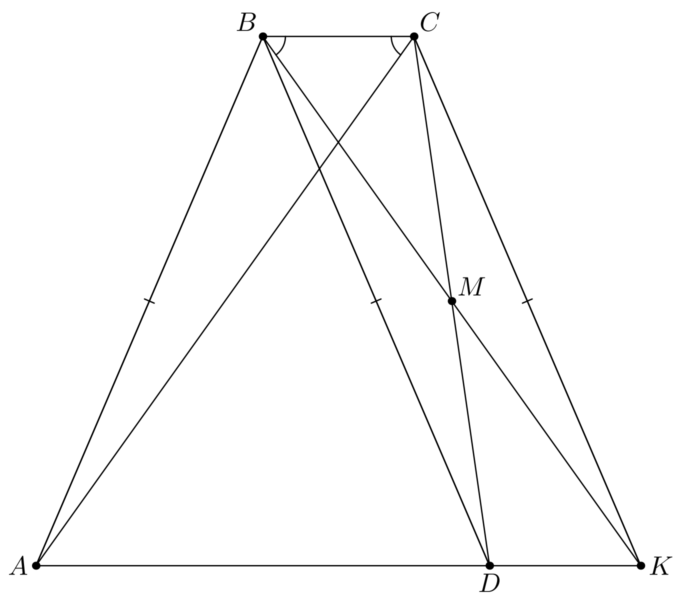

Fig.1

2. (L.Emelyanov, 8) Mark three nodes on a cellular paper so that the semiperimeter of the obtained triangle would be equal to the sum of its two smallest medians.

   Solution. Mark three vertices A, B, C of a right-angled triangle with legs $AC = 6$, $BC = 4$. Its median from C is equal to a half of hypothenuse AB, and its median from B by the Pythagorean theorem is equal to $\sqrt{BC^2 + (AC/2)^2} = \sqrt{4^2 + 3^2} = 5 = (AC + BC)/2$, hence ABC is the required triangle.

3. (E.Diomidov, 8) Let AH_1, BH_2 be two altitudes of an acute-angled triangle ABC, D be the projection of H_1 to AC, E be the projection of D to AB, F be a common point of ED and AH_1. Prove that $H_2F \parallel BC$.

   Solution. Let H be the orthocenter of triangle ABC. Using the Thales theorem we obtain (fig.3)

$$\frac{AF}{AH_1} = \frac{AF}{AH} \cdot \frac{AH}{AH_1} = \frac{AD}{AC} \cdot \frac{AH_2}{AD} = \frac{AH_2}{AC}.$$

   From this, also by the Thales theorem we obtain the required assertion.

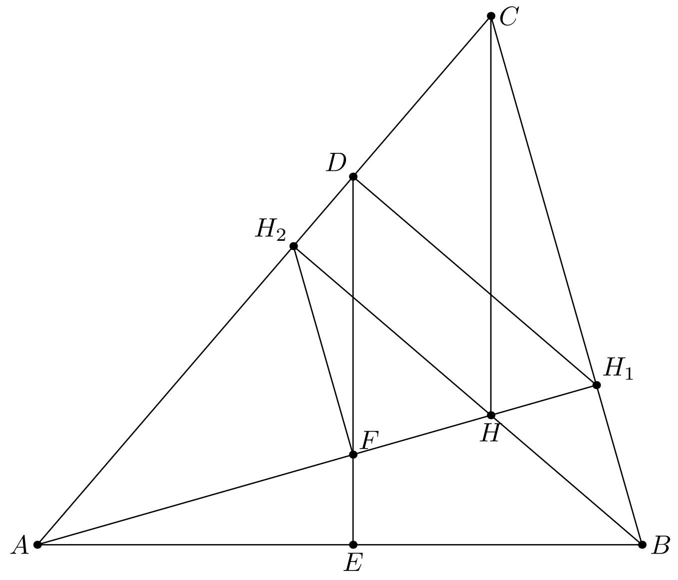

Fig.3

4. (A.Trigub, 8) In a quadrilateral $ABCD$ $\angle B = \angle D = 90^\circ$ and $AC = BC + DC$. The point P of ray BD is such that $BP = AD$. Prove that the line CP is parallel to the bisector of angle ABD.

   Solution. The assumption yields that the quadrilateral $ABCD$ is inscribed into the circle with diameter AC. Let K be a point of segment AC such that $AK{=}BC$ (fig.4). Then $CK = CD$, i.e. $\angle CKD = \angle CDK$. Now the triangles BCP and AKD are congruent because $AK = BC$, $AC = BP$ and $\angle KAD = \angle CAD = \angle CBD = \angle CBP$. Therefore $\angle BCP = \angle AKP = 180^\circ - \angle CKD = 90^\circ + \frac{\angle ACD}{2} = 90^\circ + \frac{\angle ABD}{2}$. On the other hand, $\angle CBP = 90^\circ - \angle ABD$, thus $\angle CPB = 180^\circ - \angle BCP - \angle CBP = \frac{\angle ABD}{2}$, and this yields the required assertion.

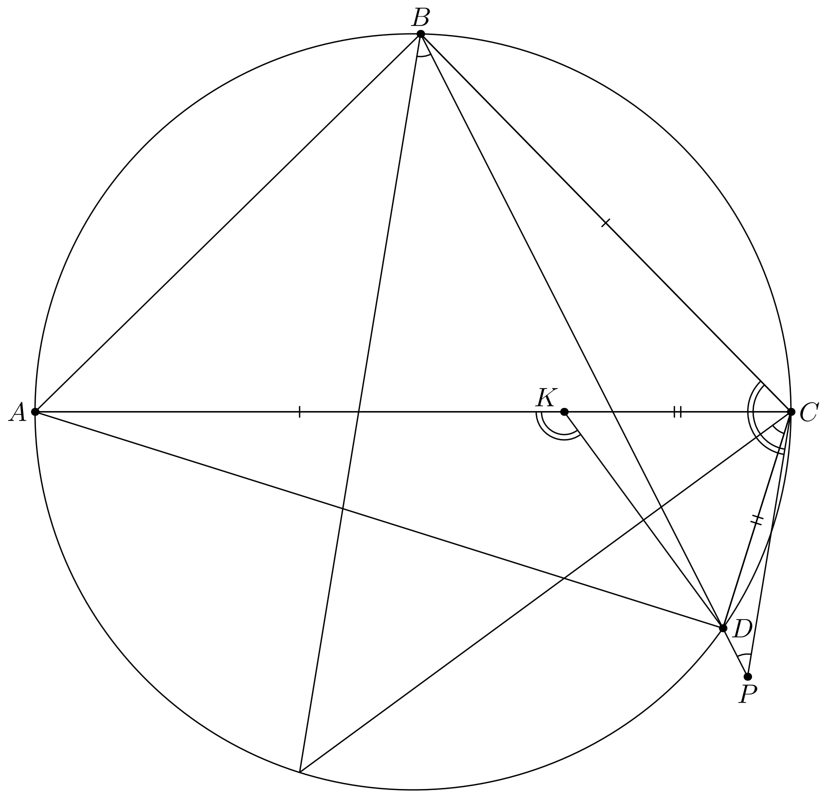

Fig.4

5. (M.Volchkevich, 8) In quadrilateral $ABCD$ $AB = CD$, M and K are the midpoints of BC and AD. Prove that the angle between MK and AC is equal to the half-sum of angles BAC and DCA.

   Solution. Construct parallelograms $ABMX$ and $DCMY$ (fig.5). Since $AX = BM = MC = DY$ and $AX \parallel BC \parallel DY$, triangles AXK and DYK are congruent. Hence $XK = KY$ and $\angle AKX = \angle DKY$, i.e. K is the midpoint of segment XY. Also we have $MX = AB = CD = MY$, therefore MK is the bisector of angle XMY, and this is equivalent to the required assertion.

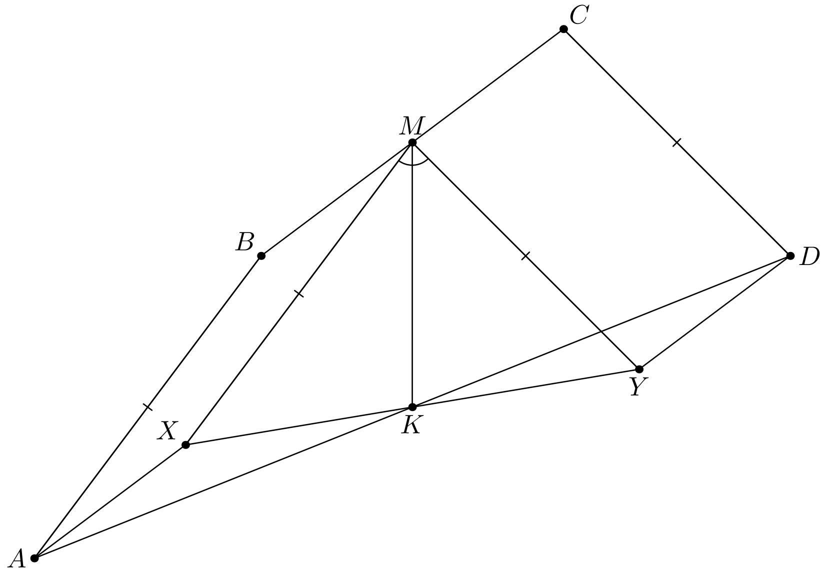

Fig.5

6. (M.Volchkevich, 8) Let M be the midpoint of side AC of triangle ABC, MD and ME be the perpendiculars from M to AB and to BC respectively. Prove that the distance between the circumcenters of triangles ABE and BCD is equal to $AC/4$.

   Solution. The segment between two circumcenters is a diagonal of the parallelogram formed by the perpendicular bisectors to segments AB, BD, BE and BC. Hence the projections of this segment to the lines AB and BC are equal to $AD/2$ and $CE/2$ respectively, i.e. they are equal to halves of the projections of segment $AM = MC$. Therefore the segment between the circumcenters is also equal to $AM/2 = AC/4$.

   Remark. From the solution we also obtain that this segment is parallel to AC.

7. (B.Frenkin, 8–9) Let all distances between the vertices of a convex n-gon ($n > 3$) be different.

   a) A vertex is called uninteresting if the closest vertex is adjacent to it. What is the minimal possible number of uninteresting vertices (for a given n)?

   b) A vertex is called unusual if the farthest vertex is adjacent to it. What is the maximal possible number of unusual vertices (for a given n)?

   Solution. a) **Answer**. 2.

   Example. Take a segment AB and a convex broken line ℓ close to it and having the same endpoints and the edges of equal length. Then ℓ and its reflection about AB form a convex polygon such that only vertices A and B are uninteresting in it. In such a way we obtain the desired n-gon for an arbitrary even $n > 2$. Now replace one of two copies of ℓ in the n-gon by an analogous broken line with the number of edges greater by 1. In this way we obtain a convex n-gon with an arbitrary odd $n > 3$, such that only the vertices A and B are uninteresting. In both cases a small shift of the vertices makes all distances between them different.

   Estimation. Let A be an interesting vertex of a convex n-gon, and B be the vertex closest to A. The diagonal AB divides the polygon into "right" and "left" parts. Let C be some vertex or right part distinct from A and B. Suppose that C is interesting and let D be the closest vertex. If D lies on the left part then in convex quadrilateral $ACBD$ we have $AB + CD < AD + CB$, i.e. the sum of the diagonals is less than the sum of two opposite sides, a contradiction. Thus D lies on the right part or on the boundary of two parts. Replacing vertices A, B to C, D we decrease the number of vertices in the right part. Since this process can not be infinite there exists an uninteresting vertex in the right part. Similarly there exists an uninteresting vertex in the left part therefore the number of uninteresting vertices is not less than two.

   b) **Answer**. 3.

   Example. Take a triangle ABC with $AB > BC > AC$. "Break" side AC a little to obtain a convex n-gon. Its unusual vertices are A, B, C only.

   Estimation. Let X be an unusual vertex, Y be the farthest vertex and Z be the vertex adjacent to Y and distinct from X. Then $XZ < XY$, hence angle XYZ is not the maximal angle of triangle XYZ and hence is acute.

Suppose that there exist more than three unusual vertices. A convex polygon has at most three acute angles. Thus there are two unusual vertices A and C for which the same vertex B is the farthest (and adjacent). Let D be an unusual vertex distinct from A, B, C and E be the farthest from it (and adjacent) vertex. Without loss of generality we can suppose that $ABED$ is a convex quadrilateral. In this quadrilateral $AB > AE, DE > BD$, i.e. the sum of the diagonals is less than the sum of two opposite sides, a contradiction.

8. (B.Frenkin, 8–9) Let $ABCDE$ be an inscribed pentagon such that $\angle B + \angle E = \angle C + \angle D$. Prove that $\angle CAD < \pi/3 < \angle A$.

   Solution. From the assumption we have $\smile AEDC + \smile ABCD = \smile BAED + \smile CBAE$, i.e. $\smile BAE = 2 \smile CD$. Since the sum of these two arcs is less than 2π, we obtain that $\smile CD < 2\pi/3$ and $\angle CAD > \pi/3$. On the other hand, since $\smile BAE < 4\pi/3$ we obtain $\smile BCDE > 2\pi/3$ and $\angle A > \pi/3$.

9. (M.Panov, 8–9) Let ABC be a right-angled triangle and CH be the altitude from its right angle C. The points O_1 and O_2 are the incenters of triangles ACH and BCH respectively; P_1 and P_2 are the touching points of their incircles with AC and BC. Prove that the lines O_1P_1 and O_2P_2 meet on AB.

   Solution. Let O_1P_1 and O_2P_2 meet AB at points K_1 and K_2. Then by Thales theorem $AK_1/K_1B = AP_1/P_1C$, $AK_2/K_2B = CP_2P_2B$. But these ratios are equal because triangles AHC and CHB are similar.

   Remark. From the solution we also obtain that the common point coincides with the touching point of the incircle of ABC with AB (fig.9).

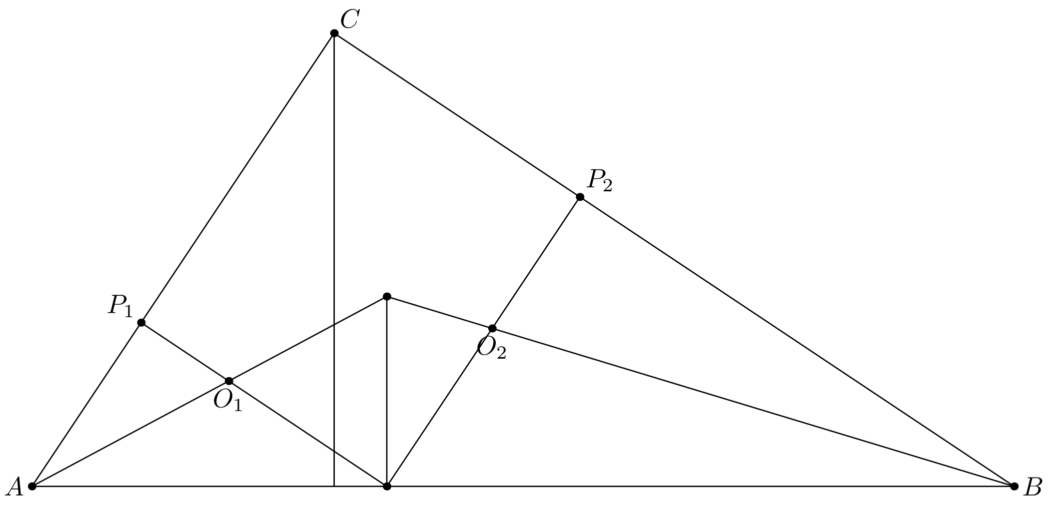

Fig.9

10. (D.Shvetsov, 8–9) The point X moves along the side AB of triangle ABC, and the point Y moves along its circumcircle in such a way that line XY passes through the midpoint of arc AB. Find the locus of the circumcenters of triangles IXY, where I is the incenter of ABC.

    Solution. Let U be the midpoint of arc AB. Since $\angle AYU = \angle ABU = \angle UAB$, triangles AUX and YUA are similar, i.e. $UX \cdot UY = UA^2$. It is known that U is the circumcenter of triangle IAB, therefore UI is a tangent to circle IXY (fig.10). Hence the center of this circle lies on the perpendicular from I to CI. Since the circle IXY cannot lie inside the circle ABC, the desired locus consists of two rays. The origins of these rays are the centers of two circles touching circle ABC internally and touching the side AB, i.e. the common points of the indicated line and the bisectors of the angles between AB and CU.

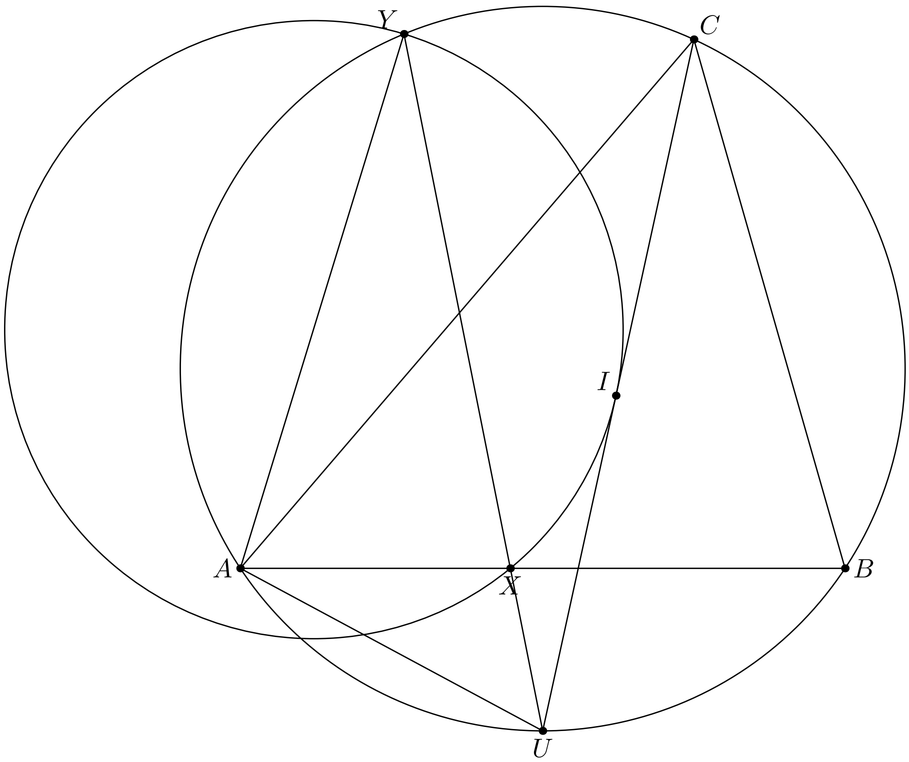

Fig.10

11. (A.Blinkov, 8–10) Restore a triangle ABC by vertex B, the centroid and the common point of the circumcircle and the symmedian going from B.

    Solution. Let the median and the symmedian from B meet the circumcircle at points K and L respectively. Since $\angle ABK = \angle CBL$, the points K and L are equidistant from the midpoint M of AC (fig.11). From this we obtain the following construction.

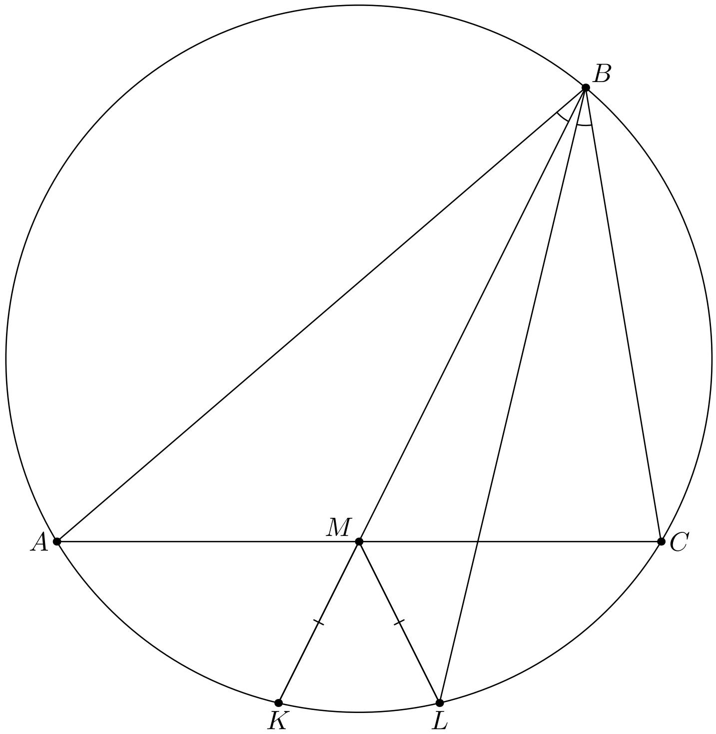

Fig.11

Extending the segment between B and the centroid by the half of its length we obtain point M. Construct the circle through L centered at M and find its common point K with BM, lying outside ray MB. Construct the circle BKL and find its common points A, C with the line passing through M and parallel to KL. The triangle ABC is the required one.

12. (S.Novikov, 9–10) Let BB_1 be the symmedian of a nonisosceles acute-angled triangle ABC. The ray BB_1 meets the circumcircle of ABC for the second time at point L. Let AH_A, BH_B, CH_C be the altitudes of triangle ABC. The ray BH_B meets the circumcircle of ABC for the second time at point T. Prove that H_A, H_C, T, L are concyclic.

    First solution. Since the points A, C, H_A, H_C are concyclic it is sufficient to prove that the lines AC, H_AH_C and TL concur. Projecting the vertices of the harmonic quadrilateral $ABCL$ from T to the line AC we obtain that the common point of TL and AC forms a harmonic quadruple with A, C, H_B. The line H_AH_C meets AC at the same point.

    Second solution. Let M be the midpoint of AC. Denote the circumcircles of triangles ABC, AHC, BH_AH_C and the circumcircle of quadrilateral AH_CH_AC by ω, ω_1, ω_2, ω_3 respectively. By the orthocenter's property the points H and T are symmetric about AC. Therefore the circles ω_1 and ω are also symmetric. Let ω_2 and ω meet for the second time at a point P, and let ω_2 and ω_1 meet for the second time at a point N.

    It is known (see. for example the paper of Y.Blinkov "The orthocenter, the midpoint of the side, the common point of the tangents and one point more", Kvant, №1, 2014) that the points M, H and P are collinear, and $\angle BPH = 90^\circ$.

    Let the lines BP and AC meet at point S. Note that H is the orthocenter of triangle BMS. Therefore $SH \perp BM$. Since $SH \perp BN$ (because $\angle BNH = 90^\circ$), we obtain that N lies on BM.

Let BM meet ω at a point D, and let the points N and N' be symmetric about AC. Since M is the midpoint of AC, and the arcs ANC and $AN'C$ are symmetric we obtain that the arcs AD and CN' of ω are equal. Line BD contains the median from B. Therefore BN' the symmedian of triangle ABC, i.e. the points N' and L coincide.

The lines NH and LT are symmetric about AC therefore they meet at S. Since S is the radical center of ω, ω_1, ω_2, ω_3 we obtain that S lies on H_CH_A (fig.12). The degrees of S wrt ω and ω_2 are equal, i.e. $SH_A \cdot SH_C = ST \cdot SL$. Therefore H_A, H_C, T, L are concyclic.

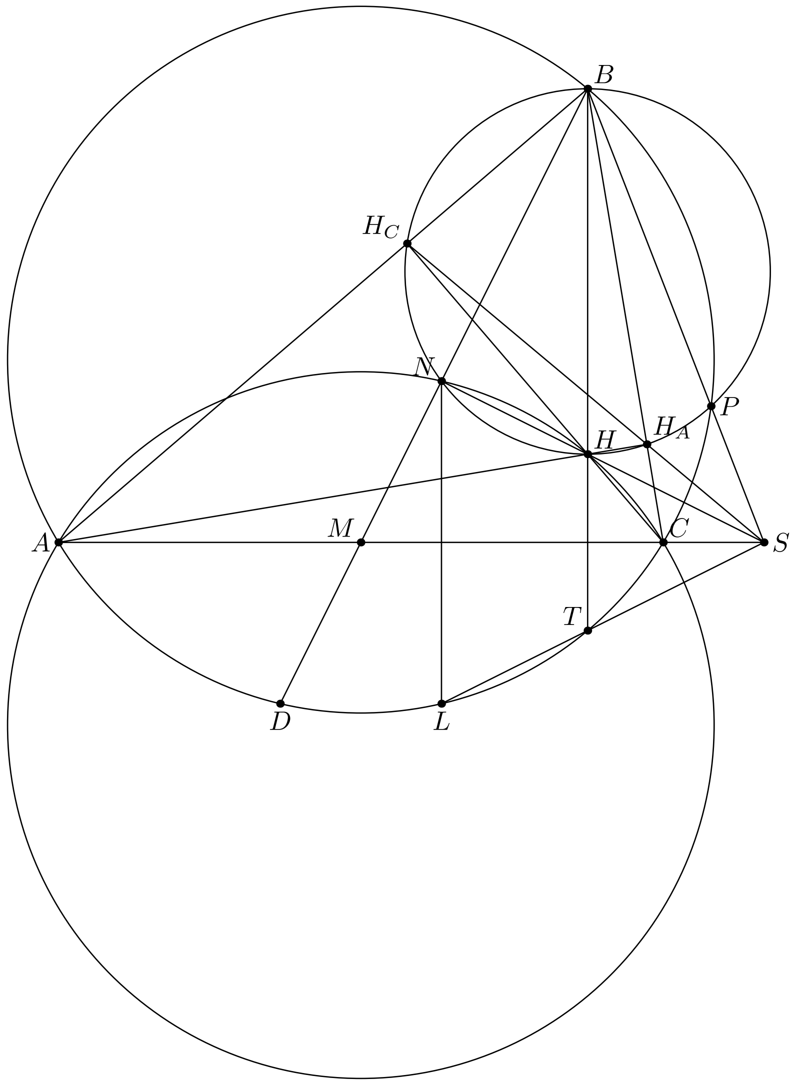

Fig.12

13. (R.Krytovsky, I.Frolov, 9–10) Given are a triangle ABC and a line ℓ meeting BC, AC, AB at points L_a, L_b, L_c respectively. The perpendicular from L_a to BC meets AB and AC at points A_B and A_C respectively. Point O_a is the circumcenter of triangle AA_bA_c. Points O_b and O_c are defined similarly. Prove that O_a, O_b and O_c are collinear.

Solution. Let Z be an arbitrary point of line AB; X, Y be the common points of the perpendicular from Z to AB with BC and CA respectively; and Z' be the circumcenter of triangle CXY. Then $\angle Z'CA = \pi/2 - \angle CXY = \angle B$, i.e. $Z'C$ touches the circumcircle of triangle ABC. If Z moves uniformly along AB then Z' also moves uniformly, and when Z coincides with A or B then Z' lies on the tangent to the circumcirle at this point. Thus if $A'B'C'$ is the triangle formed by three tangents then Z' divides segment $A'B'$ in the same ratio as Z divides AB. Applying this to points O_a, O_b, O_c and using the Menelaus theorem we obtain the required asssertion.

14. (A.Myakishev) Let a triangle ABC be given. Consider the circle touching its circumcircle at A and touching externally its incircle at some point A_1. Points B_1 and C_1 are defined similarly.

    a) (9–10) Prove that lines AA_1, BB_1 и CC_1 concur.

    b) (10–11) Let A_2 be the touching point of the incircle with BC. Prove that the lines AA_1 and AA_2 are symmetric about the bisector of angle A.

    Solution. a) Denote the first of the indicated circles by α. The point A is the center of the positive homothehy of α and the circumcircle of triangle ABC, and the point A_1 is the center of the negative homothety of α and this incircle. Therefore the line AA_1 passes through the center of the negative homothety between the incircle and the circumcircle. Two remaining lines also pass through this point.

    b) It is known that the center of the negative homothety between the incircle and the circumcircle is isogonally conjugated to the Gergonne point lying on the lines AA_2, BB_2 and CC_2. The desired assertion immediately follows from this.

15. (L.Emelyanov, 9–11) Let O, M, N be the circumcenter, the centroid and the Nagel point of a triangle. Prove that angle MON is right if and only if one of the triangle's angles is equal to $60°$.

    Solution. Let I, H be the incenter and the orthocenter respectively of the triangle. The homothety with center M and coefficient $-1/2$ maps N and H to I and O respectively. Thus $\angle MON = \pi/2$ if and only if $IO = IH$. Let the line OH intersect the segments AC and BC. Then since AI and BI are the bisectors of angles HAO and HBO, we obtain that the points A, B, O, I, H are concyclic. Therefore $\angle AOB = 2\angle C = \angle AHB = \pi - \angle C$ and $\angle C = 60°$. The inverse assertion can be proved similarly.

16. (A.Doledenok, 9–11) Let BB_1 and CC_1 be the altitudes of triangle ABC. The tangents to the circumcircle of AB_1C_1 at B_1 and C_1 meet AB and AC at points M and N respectively. Prove that the common point of circles AMN and AB_1C_1 distinct from A lies on the Euler line of ABC.

    Solution. Let A_0, B_0, C_0 be the midpoints of BC, CA, AB; O, H be the circumcenter and the orthocenter of triangle ABC. The projection Z of A to line OH lies on circles AB_1HC_1 and AB_0OC_0, i.e., Z is the center of the spiral similarity mapping C_0 to B_0, and C_1 to B_1. Thus if we prove that this similarity maps M to N we obtain that circle AMN passes through Z.

    Note that point A_0 and the center of circle AB_1HC_1 are opposite on the nine points circle of triangle ABC. Hence lines A_0B_1 and A_0C_1 are tangents to circle AB_1HC_1, i.e. they coincide with lines B_1M and C_1N (fig.16). Projecting line AC to AB from point A_0 we obtain that $(N, B_1, B_0, \infty) = (C_1, M, \infty, C_0)$ or $NB_0/NB_1 = MC_0/MC_1$, q.e.d.

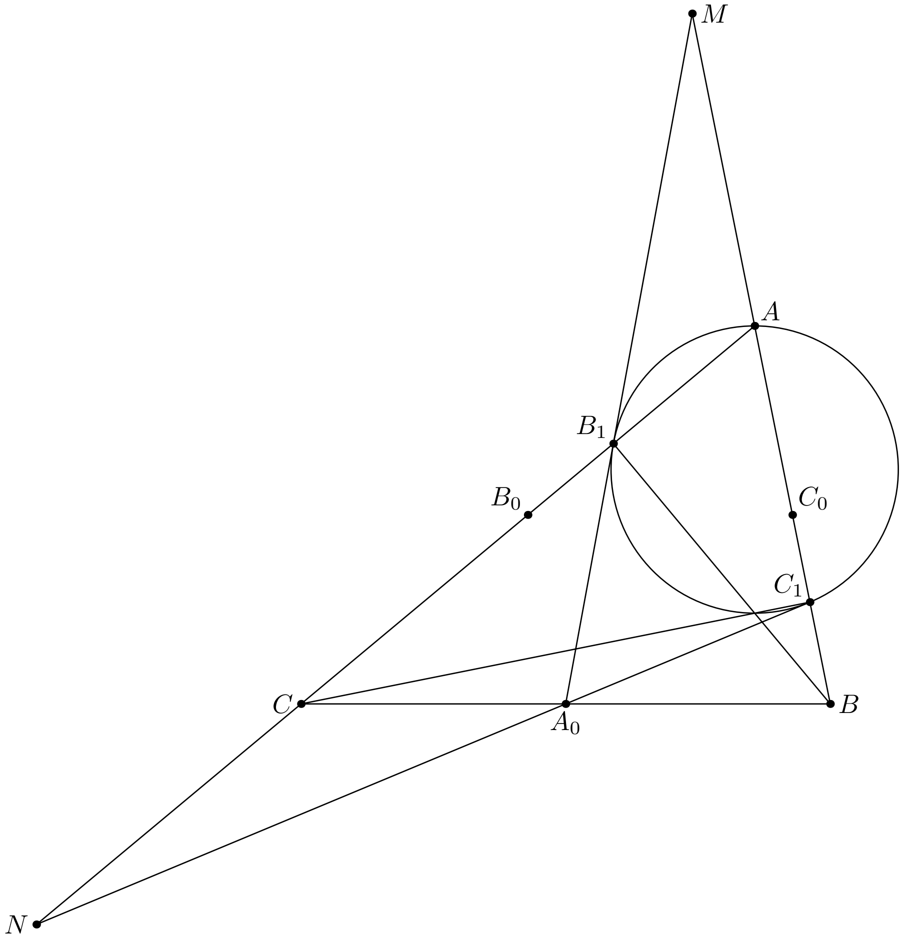

Fig.16

17. (D.Hilko, 9–11) Let D be an arbitrary point on the side BC of triangle ABC. The circles ω_1 and ω_2 pass through A and D in such a way that BA touches ω_1 and CA touches ω_2. Let BX be the second tangent from B to ω_1, and CY be the second tangent from C to ω_2. Prove that the circumcircle of triangle XDY touches BC.

    Solution. Take an inversion with center D and an arbitrary radius. Denote the images of all points by primes (fig.17).

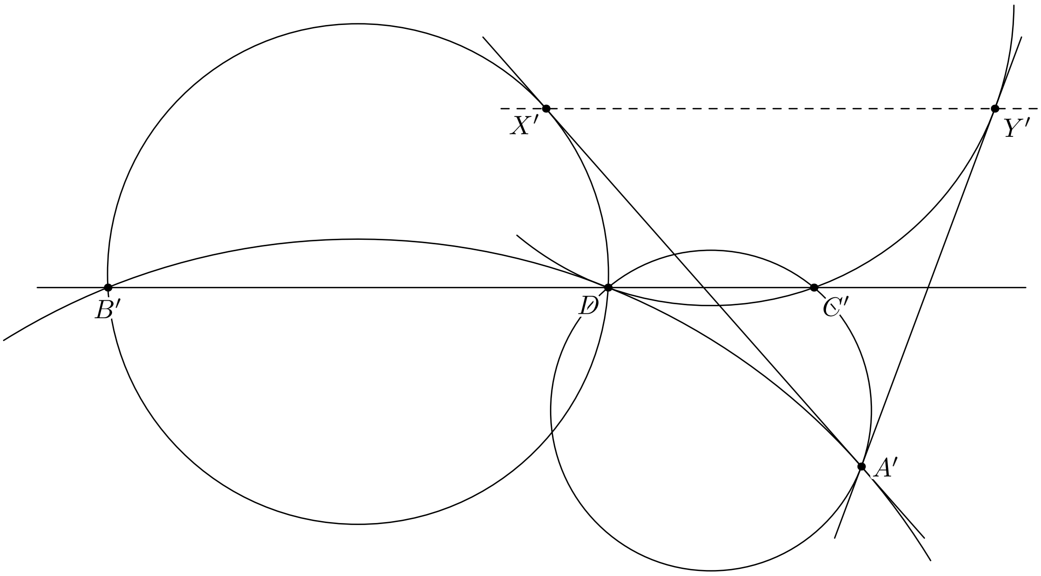

Fig.17

The circumcircle of triangle XDA touches BA and BX. Therefore the circumcircles of triangles $B'DA'$ and $B'DX'$ touch line $X'A'$. Then the radical axis $B'D$ of these circles bisects segment $X'A'$. Similarly circles $DC'Y'$ and $DC'A'$ touch line $Y'A'$. Then their radical axis DC' bisects segment $A'Y'$. Hence $B'C'$ is the medial line of triangle $X'A'Y'$ and $X'Y' \parallel B'D'$. Observe now that $X'Y'$ is the image of the circle passing through X, Y, D. Since $X'Y' \parallel B'C'$ this circle touches BC at point E.

18. (N.Moskvitin, 9–11) Let ABC be a triangle with $\angle C = 90°$, and K, L be the midpoints of the minor arcs AC and BC of its circumcircle. The segment KL meets AC at point N. Find angle NIC where I is the incenter of ABC.

    Answer. 45°.

    Solution. It is known that points K and P are the circumcenters of triangles IAC and IBC respectively. Thus KP is the perpendicular bisector for segment CI. Then N is the touching point of AC with the incircle and $\angle NIC = 45°$ (fig.18).

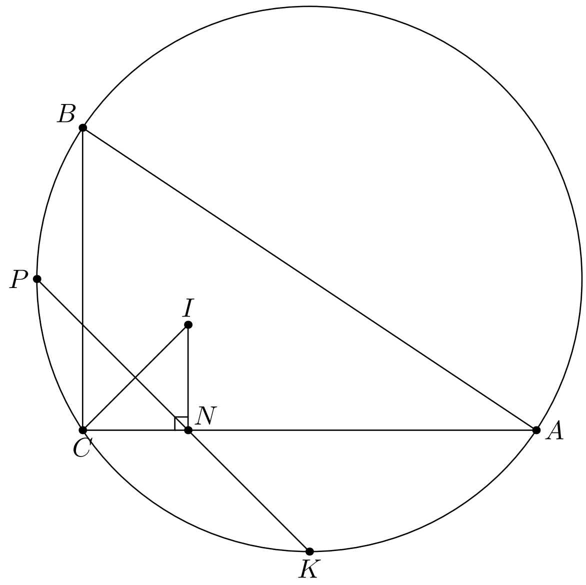

Fig.18

19. (A.Skutin, 9–11) Let $ABCDEF$ be a regular hexagon. The points P and Q on tangents to its circumcircle at A and D respectively are such that PQ touches the minor arc EF of this circle. Find the angle between PB and QC.

Answer. 30°.

Solution. Let T be the touching point of PQ with the circle and M, N be the midpoints of segments AT, DT. Since PB and CQ are the symmedians of the triangles ABT, CDT respectively, we have $\angle ABP = \angle MBT$, $\angle DCQ = \angle NCT$. Since MN is the medial line of triangle ADT, we have $MN = AD/2 = BC$ and $MN \parallel BC$ (fig.19). Thus the angle between PB and QC is equal to $\angle PBM + \angle NCQ = \angle ABM + \angle NCD - \angle MBT - \angle TCN = 30^\circ$.

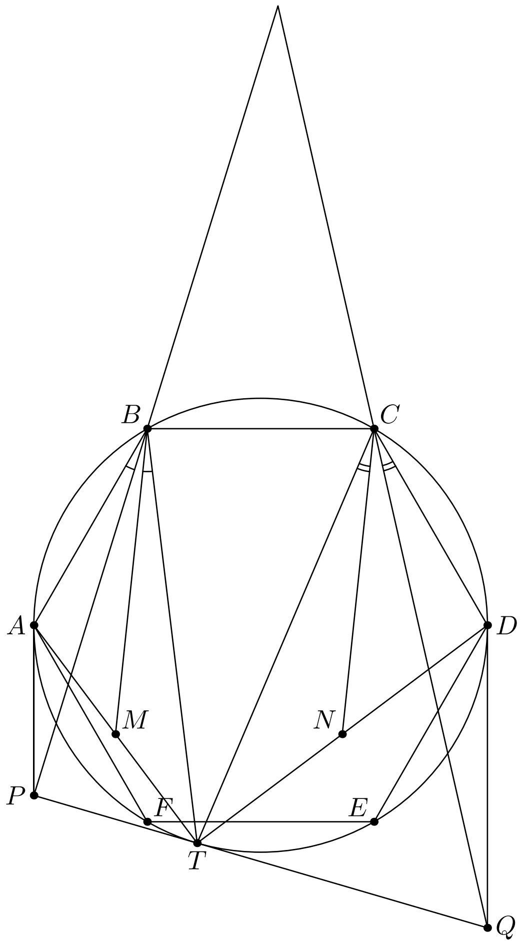

Fig.19

20. (D.Prokopenko, 10–11) The incircle ω of a triangle ABC touches BC, AC and AB at points A_0, B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to segment AA_0 at points Q and P respectively. Prove that PC_0 and QB_0 meet on ω.

Solution. The definition of points P, Q implies that they lie on the circumcircles of triangles ABA_0 and ACA_0 respectively. Therefore triangle AA_0Q is similar to triangle B_0A_0I, and triangle AA_0P is similar to triangle C_0A_0I (by three angles). Thus $A_0Q \cdot A_0B_0 = A_0I \cdot A_0A = A_0P \cdot A_0C_0$. Furthermore $\angle PA_0Q = (\angle B + \angle C)/2 = \angle B_0A_0C_0$, hence triangles A_0PQ and $A_0B_0C_0$ are similar (fig.20). Then triangles A_0B_0P and A_0C_0Q also are similar, i.e. the angle between B_0P and C_0Q is equal to angle $B_0A_0C_0$, and this yields the required assertion.

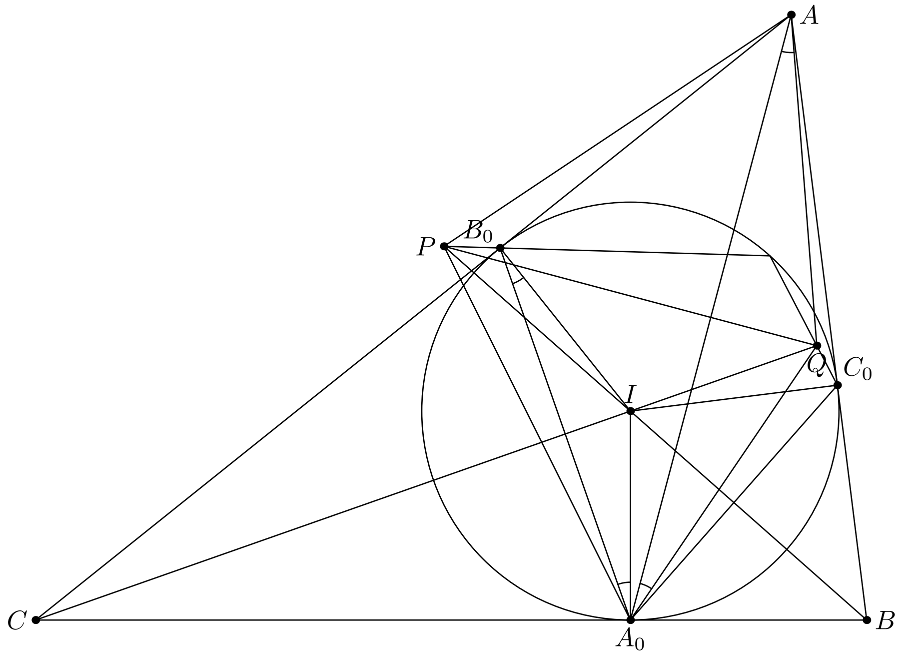

Fig.20

21. (A.Shapovalov, 10–11) The areas of rectangles P and Q are equal, but the diagonal of P is greater. Rectangle Q can be covered by two copies of P. Prove that P can be covered by two copies of Q.

    Solution. Let the width and the length of a rectangle mean its smaller and greater side respectively. From the assumption we have that the width of P is less than the width of Q, and the length of P is greater than the length of Q. If two copies of P cover Q, then they cover the disc with the diameter equal to the width of Q, therefore this disc can be covered by two bars with the width equal to the width of P. But the disc cannot be covered by bars if the sum of their widths is less than the diameter. Thus the width of P is at least the half of the width of Q. Then the length of Q is at least the half of the length of P, and clearly P can be covered by two copies of Q.

22. (A.Yakubov, 10–11) Let M_A, M_B, M_C be the midpoints of the sides of a nonisosceles triangle ABC. The points H_A, H_B, H_C lying on the corresponding sides and distinct from M_A, M_B, M_C are such that $M_AH_B = M_AH_C$, $M_BH_A = M_BH_C$, $M_CH_A = M_CH_B$. Prove that H_A, H_B, H_C are the bases of the altitudes of ABC.

    Solution. Consider a point X in the space such that $XM_A = M_AH_B$, $XM_B = M_BH_A$, $XM_C = M_CH_A$. Consider tetrahedron $XM_AM_BM_C$. The areas of all its faces are equal because triangles XM_AM_B and $H_CM_AM_B$ are congruent. Hence all faces are congruent and points H_C, M_A, M_B, M_C are concyclic. Therefore H_C is the base of the altitude.

23. (F.Ivlev, 10–11) A sphere touches all edges of a tetrahedron. Let a, b, c and d be the segments of the tangents to the sphere from the vertices of the tetrahedron. Is it true that some of these segments necessarily form a triangle? (Not all those segments must be used. Two segments may form one side of the triangle.)

    Answer. No.

Solution. Let β and γ be circles of radii 2 and 1 respectively that lie in the plane and touch externally. Construct their common external tangent and inscribe circle δ into the curvilinear triangle formed by two circles and this tangent. Clearly the radius of δ is less than 1, so the radii of three circles do not form a triangle. Now replace the common tangent of β and γ by circle α touching them externally with radius greater than 4. Construct three spheres with the same centers and radii as α, β, γ. Finally construct the sphere having the same radius as δ and touching three remaining spheres. The centers of these four spheres form a tetrahedron, and their touching points lie on the sphere touching all edges of this tetrahedron. The segments a, b, c, d are equal to the radii of α, β, γ, δ, therefore they don't form a triangle.

24. (I.I.Bogdanov, 11) A sphere is inscribed into a prism $ABCA'B'C'$ and touches its lateral faces $BCC'B'$, $CAA'C'$, $ABB'A'$ at points A_0, B_0, C_0 respectively. It is known that $\angle A_0BB' = \angle B_0CC' = \angle C_0AA'$.

a) Find all possible values of these angles.

b) Prove that segments AA_0, BB_0, CC_0 concur.

c) Prove that the projections of the incenter to $A'B'$, $B'C'$, $C'A'$ are the vertices of a regular triangle.

Solution. a) **Answer.** 60°.

Denote the value of these angles by θ. Since the triangles $CC'A_0$ and $CC'B_0$ are congruent we obtain that the angle A_0CC' is also equal to θ. Similarly $\angle B_0AA' = \angle C_0BB' = \theta$. Then $6\theta = 3\pi - (\angle C_0AB + \angle C_0AC + \angle A_0BC + \angle A_0CB + \angle B_0CA + \angle B_0AC)$. But for example $\angle C_0AB = \angle TAB$, where T is the touching point of the sphere with face ABC. From this and five similar equalities we obtain that the sum in the parentheses is equal to the sum of the angles of triangle ABC, i.e. $\theta = 60^\circ$.

b) By the previous part, $\angle AB_0C = \angle BA_0C = 2\pi/3$. Thus the lines AB_0 and BA_0 meet CC' at the same point K such that $CK = CB_0 = CA_0$ (the triangles CB_0K and CA_0K are regular because each of them has two angles equal to $\pi/3$). Therefore the points A, B, A_0, B_0 are complanar, i.e. the lines AA_0 and BB_0 intersect. Similarly each of these lines intersects CC_0. Since these three lines are not complanar the points of intersection coincide.

c) By the previous part, $\angle ATB = \angle BTC = \angle CTA = 2\pi/3$, i.e. T is the Torricelli point of triangle ABC. Consider another sphere touching the plane ABC from the opposite side at point T' and touching the planes of lateral faces. The ratios of distances from T and T' to the sidelines of ABC are equal to the ratios of the cotangents and tangents of the halves of the corresponding dihedral angles, therefore these points are isogonally conjugate in triangle ABC. Hence the insphere touches the face $A'B'C'$ at its Apollonius point. The projections of this point to $A'B'$, $B'C'$, $C'A'$ coincide with the projection of the center of the sphere and form a regular triangle.