2013 (English)

Regatta

Junior League

First Round

1a. There is a heap of sweets. First Lillebror eats one of the sweets, then Karlsson eats two 2 sweets, then Lillebror eats three, Karlsson eats four, and so on. If the heap contains less sweets than someone (a boy or Karlsson) has to eat next turn, then he eats the rest part of sweets from the heap. It turned out that Lillebror ate 101 sweets. How many sweets were in the heap at the beginning of a meal?

(А. Шаповалов)

1g. Points M, N, P are midpoints of sides AB, CD and DA of inscribed quadrilateral $ABCD$. Given that $\angle MPD = 150^\circ$, $\angle BCD = 140^\circ$, find $\angle PND$.

(Д. Максимов)

1c. Three brothers have to move from one apartment to another a piano (250 kg), a sofa (100 kg) and more than 100 boxes (50 kg each). They hired a small van with a driver to do 5 trips (and 4 back trips); the van can carry 500 kg of cargo and one passenger. A sofa can be loaded or unloaded by two brothers, piano by three brothers, each box can be loaded or unloaded by any brother. It is necessary to transport all the furniture and as many as possible boxes. What is the maximum number of boxes that can be transported? (The driver can not load or unload goods, there is no another transport, passengers cannot be taken instead of cargo).

(А. Шаповалов, Д. Шаповалов)

Second Round

2a. 2013 positive integers are arranged in a circle. In each pair of adjacent numbers one number is divisible by the other. Prove that there are two non-adjacent numbers, one of which is also divided by the other.

(А. Шаповалов)

2g. There are 2013 unit segments on the plane. Any two segments have an intersection point. Prove that all these segments can be covered by a disk of radius 1.5.

(А. Шаповалов)

2c. The factory produces sets of $n > 2$ elephants of different sizes and masses. The inspector checks each set separately. As a standard, when the set is arranged so that masses go in ascending order, the mass difference in each pair of adjacent elephants should be the same. The inspector checks the sets with scales without weights. What is the least possible n such that this test can be organized correctly?

(А. Шаповалов)

Third Round

3a. Vasya has written numbers from 1 to 20132013 on a blackboard. Let a be the number of digits 1 on blackboard and b be the number of digits 3. What is the difference between a and b?

(folklore, proposed by Д. Максимов)

3g. In isosceles trapezoid $ABCD$ inscribed a circle with center O. A point M is a middle of a longer base AB. Line MO intersects the segment CD at the point F. E is the point of tangency CD and the circle. Prove that $DE = CF$ if and only if $AB = 2CD$.

(Polish NO 1993)

3c. Chess board is covered with 32 dominoes (each domino covers exactly two fields). Prove that dominoes can be rotated by 90 or 180 degrees (each around the center of one of covered cells, they can be rotated independently), so that the entire board will be covered again.

(А. Шаповалов)

Fourth Round

4a. A square table 100×100 is filled with positive integers so that all 200 sums in rows and columns are different. What is the least possible value of total sum of numbers in the table?

(Д. Максимов)

4g. Given a parallelogram $ABCD$ with $BD = BC$. The point M on the segment AC is such that $3AM = AC$. Prove that $AM = BM$.

(13-й Уральский турнир, высшая лига)

4c. The stair consists of 350 steps. There are 350 stones on the bottom step of a stair. All other steps are empty. Each move Sisyphus can take from any step a group (one or more) of stones (not necessarily all stones from the step), and shift the entire group up or down by the number of steps equal to the number of stones in the group (a group of one stone to the next or previous step, a group of two stones by two step up or down, etc.). Help Sisyphus shift all the stones to the next step with no more than 21 moves.

(А. Шаповалов)

Senior League

First Round

1a. The number is *interesting* if it is a power of three or can be represented as the sum of different powers of three. Interesting numbers are numbered in ascending order. Find the hundredth number.

(folklore)

1g. Points M and N are chosen on a hypotenuse AC of rectangular isosceles triangle ABC. Given that $\angle MBN = 45°$, prove that it is possible to construct a rectangular triangle from the segments AM, MN and NC.

(folklore)

1c. A closed simple (self-avoiding) polygonal chain consists of 8 segments. Its vertices coincide with the vertices of a cube. Prove that given chain has four segments of equal length.

(А. Шаповалов)

Second Round

2a. What is a total amount of positive integers N from 1 to 2013 such that the equation $x^{[x]} = N$ has a solution? ($[x]$ is the largest integer less or equal than x.)

(folklore)

2g. Given a point Q inside a convex polyhedron M. A line ℓ passes through Q and intersects the surface of M at points A and B. Prove that for infinitely many of lines ℓ the equality $AQ = BQ$ holds.

(Putnam 1977 B4 Problem)

2c. There are $4n^2$ kings on a chessboard $4n \times 4n$ that do not beat each other. Prove that the number of such arrangements of kings does not exceed 12^{2n^2}.

(А. Шаповалов)

Third Round

3a. Given a quadratic polynomial $ax^2 + bx + c$ with $a \neq 0$. Prove that for some irrational x the value $ax^2 + bx + c$ is rational.

(С. Когаловский, А. Шаповалов)

3g. A quadrilateral $ABCD$ is inscribed in a circle. K, L, M and N are chosen on the segments AB, BC, CD and DA respectively in such a way that $AK = KB = 6$, $BL = 3$, $LC = 12$, $CM = 4$, $MD = 9$, $DN = 18$, $NA = 2$. Prove that quadrilateral $KLMN$ can be inscribed into a circle.

(Ф. Бахарев)

3c. Lets call a digit of a number *valuable* if it occurs in decimal expansion not more times than each of the other digits (it may not occur at all). Peter generates an infinite sequence of numbers: he puts any of the valuable digits of the number n at the n-th place. Can Peter obtain a sequence which is periodical from some point?

(А. Шаповалов)

Fourth Round

4a. A base of a right prism is a right triangle. All edges of the prism have integer lengths. Areas of two faces of the prism are equal to 13 and 30. Find sides of triangle base of the prism.

(KöMaL journal)

4g. A nonisosceles triangle ABC is inscribed in a circle Γ. The bisector of angle A meets BC at E. Let M be the midpoint of the arc BAC. The line ME intersects Γ again at D. Show that the circumcentre of triangle AED coincides with the intersection point of the tangent to Γ at D and the line BC.

(Italy TST 2002)

4c. There are $2n^2 + 5n$ positive integers written on a blackboard (not necessarily distinct). Each number is equal to the total amount of numbers which are not equal to this number. What is the largest possible number of different numbers on the board?

(А. Шаповалов)

Algebra and Number Theory

Junior League

1. Find all positive integers n and k such that $1! + 3! + \ldots + (2n-1)! = k^2$.

(KöMaL journal)

2. Positive numbers a, b and c satisfy the equation $a^2 + b^2 + c^2 + 2abc = 1$. Prove the inequality

$$a\sqrt{(1-b^2)(1-c^2)} + b\sqrt{(1-c^2)(1-a^2)} + c\sqrt{(1-a^2)(1-b^2)} \geqslant 2\sqrt{abc}.$$

(Crux Mathematicorum)

3. Find all positive integers a и b such that $\sqrt{a} - \sqrt{b}$ is a root of the equation $x^2 + ax - b = 0$.

(Д. Максимов)

4. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) + f(n) = 2n + 6$ for all $n \in \mathbb{N}$. ($\mathbb{N}$ is a set of positive integers.)

(Д. Максимов)

Senior League

1. A square table 100×100 is filled with positive integers so that all 200 sums in rows and columns are different. What is the least possible value of total sum of numbers in the table?

(Д. Максимов)

2. An increasing function $f\colon (0;\infty) \to (0;\infty)$ satisfies the inequality $f(2x) \leqslant 4f(x+1)$ for all $x > 0$. Prove that for every $x > 1$

$$f(x) \leqslant f(4)x^2.$$

(В. Быковский)

3. Let

$$x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4,$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2,$$
$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3.$$

Prove that

$$(x_1 - y_2)(x_1 - y_3)(x_1 - y_4) = (y_1 - x_2)(y_1 - x_3)(y_1 - x_4).$$

(А. Устинов)

4. Is it possible to find a sequence of 1000000 positive integers coprime to 10 such that each number is divisible by previous one but its sum of digits is smaller?

(А. Шаповалов)

Combinatorics and Logic

Junior League

1. 60 students participated in a school mathematics contest, 50 in a physics contest and 40 in a computer science contest. One draw up three lists: the first list is of those who took part in exactly one competition, second list is of those who took part in exaclty two competitions, and third list is of those who took part in all three competitions. All lists contain the same number of items. How many items are there in each list?

(А. Шаповалов)

2. The center of regular 55-sided polygon is connected with all its vertices by segments. Entire picture (together with the sides of polygon) contains 110 segments. Peter and Bob colour these segments, one per turn. One can colour a segment AB if this segment and and all the other segments incident to A or B are not coloured. A person who cannot make a move loses the game. Peter begins. Who wins, regardless of the opponent's moves?

(А. Шаповалов)

3. There are 77 knights, 33 rooks and queen on 100×100 chessboard. Each chess piece is able to capture exactly one other piece and can be captured exactly by one other piece. Prove that the queen is able to capture a piece diagonally.

(А. Шаповалов)

4. 33 computers are pairwise connected by wires. Each wire is painted in one of 32 colors. Computer is called *rainbowed* if it is connected with other computers with wires of all 32 colors. What is the largest possible number of rainbowed computers?

(Iran NO 2012)

5. A few weights are arranged in a circle. They are not all the same mass, and their masses are not necessarily integer. In each consequtive triple of weights, the mass of one of them is equal to the arithmetic mean of all three masses. Prove that the number of weights is divisible by 3.

(А. Шаповалов)

Senior League

1. Peter and Bob colour the edges of a pyramid with an 77-sided base, one per turn. One can colour an edge AB if this edge and all the other edges incident to A and B are not coloured. A person who can not make a move loses the game. Peter begins. Who wins, regardless of the opponent's moves?

(А. Шаповалов)

2. There are 2013 chess pieces on the infinite chessboard: knights, rooks and queens. Each chess piece is able to capture exactly one another piece and can be captured exactly by one another piece. Prove that there is the queen that is able to capture a piece diagonally.

(А. Шаповалов)

3. A rectangular board is divided into 99 rows and 99 columns by lines parallel to its sides (so that there are 99^2 rectangular cells). Some of the cells are marked. There are no two equal marked rectangles, and each unmarked rectangle is equal to some marked one. Is it possible that there are exactly 1999 marked rectangles?

(А. Шаповалов)

4. A few weights are arranged in a circle. They are not all the same mass, and their masses are not necessarily integer. In each consequtive triple of weights, the mass of one of them is equal to the arithmetic mean of all three masses. Prove that the number of weights is divisible by 3.

(А. Шаповалов)

5. From the deck of 52 cards the ace of spades is lost. For each pair of cards of the same suit or of the same value the number of cards between them is known. Prove that this information is enough to determine a pair of the topmost and the bottommost card of the deck.

(А. Шаповалов)

Geometry

Junior League

1. Points D on the side AC and E on the side BC of triangle ABC are such that $\angle ABD = \angle CBD = \angle CAE$ and $\angle ACB = \angle BAE$. Let F be the intersection point of segments BD and AE. Prove that $AF = DE$.

(Ф. Ивлев, Ф. Бахарев)

2. A hexagon $ABCDEF$ is inscribed into a circle. X is the intersection point of the segments AD and BE, Y is the intersection of AD and CF, and Z is the intersection of BE and CF. Given $AX = DY$ and $CY = FZ$, prove that $BX = EZ$.

(Д. Максимов, Ф. Петров)

3. A quadrilateral $ABCD$ is inscribed into a circle, given $AB > CD$ and $BC > AD$. Points K and M are chosen on the rays AB and CD respectively in such a way that $AK = CM = \frac{1}{2}(AB + CD)$. Points L and N are chosen on the rays BC and DA respectively in such a way that $BL = DN = \frac{1}{2}(BC + AD)$. Prove that the $KLMN$ is a rectangle of the same area as $ABCD$.

(Eisso J. Atzema, proposed by В. Дубровский)

4. Point O is the circumcenter and point H is the orthocenter in an acute nonisosceles triangle ABC. Circle ω_A is symmetric to the circumcircle of AOH with respect to AO. Circles ω_B and ω_C are defined similarly. Prove that circles ω_A, ω_B and ω_C have a common point, which lies on the circumcircle of ABC.

(Ф. Бахарев, inspired by Iran TST 2013)

Senior League

1. AA_1 and BB_1 are the altitudes in an acute triangle ABC, and O is its circumcenter. Prove that the areas of the triangles AOB_1 and BOA_1 are equal.

(folklore)

2. Two circles ω and γ have the same center, and γ lies inside ω. Let O be an arbitrary point on ω. OA and OB are the tangent lines through O to γ. Circle with center O and radius OA meets ω at points C and D. Prove that the line CD contains the middle line of the triangle OAB.

(Ф. Ивлев)

3. For which $a > 1$ there exists a convex polyhedron such that the ratio of areas of any two of its faces is greater than a?

(А. Шаповалов)

4. Let H, I, O be the orthocenter, incenter and circumcenter of an acute triangle ABC respectively. Prove that $\angle AIH = 90^\circ$ if and only if $OI \parallel BC$.

(Ф. Ивлев)

Team Contest

Junior League

1. (3) A family consists of the father, mother and two children. They call each other by their names. In a conversation between a parent and a child, all referred persons are listed from junior to senior, and in conversations between children or between parents persons are always listed in the opposite order. Alex said to Sveta: «We are going to the theater: Nastya, me and Volodya». What are the names of mother and father?

(А. Шаповалов)

2. (4) All participants of a chess tournament are enumerated. Each pair of players plays one game. In all games that did not end by draw, the number of the winner is less than the number of the loser. Peter defeated Vasya, but Peter's final score occured less than Vasya's one. What is the least possible number of players in the tournament? (Players get 1 point for a win, 0.5 points for a draw and 0 points for a loss.)

(А. Шаповалов)

3. (5) Given positive integers m and n such that $4^m - 1$ is divisible by n, and $n - 1$ is divisible by 2^m, prove that $n = 2^m + 1$.

(Indonesia NO 2012)

4. (6) Given the isosceles triangle ABC ($AB = BC$). On the extension of the side AB past B is chosen point D, such that $\angle DCB = \angle BCA$. On the altitude BH of triangle ABC is chosen the point E, such that $DE = DC$. Prove that $\angle BDE = \frac{1}{3}\angle BDC$.

(Georgia TST 2005)

5. (6) Inside the triangle ABC is chosen point P such that $\angle ABP = \angle CPM$, where M is a middle of the segment AC. The line MP intersect circumcircle of the triangle APB at points P and Q Prove that $QA = PC$.

6. (7) For the sequence of positive integers u_0, u_1, ... we know that $u_0 = 1$ and for some positive ineger k holds recurrent relation $u_{n+1}u_{n-1} = ku_n$ ($n \geqslant 0$). Find k if $u_{2013} = 100$.

(British Olympiad 1995)

7. (8) Three runners are training on a straight track. They have different (constant) speeds. When any of them reaches an end of the track, he immediately turns around and runs back, then turns around at the other end of the track, and so on. It occured that five times all the runners found themselves at one point. Prove that there will be infinite number of such meetings.

(А. Шаповалов)

8. (10) Positive reals x, y and z satisfy the condition $x + y + z = 1$. Prove that

$$\frac{xy}{\sqrt{z+xy}} + \frac{yz}{\sqrt{x+yz}} + \frac{zx}{\sqrt{y+zx}} \leqslant \frac{1}{2}.$$

(Д. Максимов)

9. (11) Is it possible to paint all points of the plane in 2013 colors (that is, to assign each point of the plane an integer from 1 to 2013), so that on any line and any circle (of non-zero radius) there will be points of all the colors?

(Jury)

Senior League

1. (3) Two medians divide a triangle into a quadrilateral and three isosceles triangles. Prove that initial triangle is also isosceles.

(А. Шаповалов)

2. (4) Are there 2013 nonzero real numbers x_1, x_2, …, x_{2013} such that for infinitely many positive integers n the formula $x_1^n + x_2^n + \ldots + x_{2013}^n = 0$ holds?

(А. Устинов)

3. (4) There is a cube with sides equal to 1 m and a set of five colors. At first, Peter cuts the cube into equal smaller cubes of size at most 1 cm (he can choose the size). Then Bob colors small cubes as he wants (not necessarily in the same way), but each face must be one-colored. Finally, Peter combines small cubes into original cube. Show that Peter (regardless Bob's coloring) can ensure that each face of large cube will be one-colored.

(А. Шаповалов)

4. (4) Let $f\colon \mathbb{N} \to \mathbb{N}$ be nondecreasing *multiplicative function* (i. e. $f(1) = 1$ and for all relatively prime positive integers m and n holds that $f(mn) = f(m)f(n)$). Prove that $f(8)f(13) \geqslant (f(10))^2$.

(folklore)

5. (5) Three runners are training on a straight track. They have different (constant) speeds. When any of them reaches an end of the track, he immediately turns around and runs back, then turns around at the other end of the track, and so on. It occured that five times all the runners found themselves at one point. Prove that there will be infinite number of such meetings.

(А. Шаповалов)

6. (6) Let $p > 3$ be a prime number. Prove that the following sum is divisible by p:

$$\left(\sum_{2 \leqslant i < j < k \leqslant p-1} ijk \right) + 1.$$

(Indonesia NO 2013)

7. (6) On the sides AB, BC, CD and DA of the rhomb $ABCD$, respectively, are chosen points E, F, G and H so, that the segments EF and GH touch the incircle of the rhomb. Prove that the lines EH and FG are parallel.

(Georgia TST 2005)

8. (8) In a given graph degrees of all the vertices are not bigger than 7. All vertices of the given grapf can't be painted in 6 colors correctly (connected vertices must have different colors). Prove that there are 3 vertices pairwise connected.

(Г. Ненашев)

9. (9) Prove that in acute-angled triangle the sum of its medians not greater than the sum of radii of its excircles.

(Ф. Ивлев)

10. (10) For integers a_1, a_2, $\ldots$, a_r greater than 1, let us define $[a_1, a_2, \ldots, a_r]$ as

$$\cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_r}}}}.$$

Given two finite sequences a_1, a_2, $\ldots$, a_n and b_1, b_1, $\ldots$, b_m of integers greater than 1 such that

$$[a_1, a_2, \ldots, a_n] + [b_1, b_2, \ldots, b_m] \geqslant 1,$$

prove that there exist indices n_1 and m_1 ($1 \leqslant n_1 \leqslant n$ and $1 \leqslant m_1 \leqslant m$) such that

$$[a_1, a_2, \ldots, a_{n_1}] + [b_1, b_2, \ldots, b_{m_1}] = 1.$$

(А. Устинов)