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Length
Area
Volume

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Preface

This small book deals with various questions related to the calculation of areas and volumes.

The main text of the book consists of four lessons.

It is expedient to develop a young student's understanding of dimension, i.e., the understanding of how the numerical characteristics (volumes and areas of geometric figures) change as the result of proportional changes of their size, before teaching how these characteristics can be formally computed by means of standard formulas (ordinarily used without explaining what is actually going on). In the first part of the book, we try to achieve this: in the *first lesson*, we discuss how area and volume change under rescaling, in the *second one*, how rescaling affects the area of a surface and, more generally, what rescaling does to combinations of different dimensions appearing in one and the same problem. These lessons are intended for students of grades 6 to 8 (ages 12-14).

Dimensional considerations suffice to understand how the volume, say, of the ball depends on its radius. But they do not suffice to find the exact value of this volume. What helps here is the layer by layer consideration of the volume in the picture — more precisely, the *Cavalieri's principle.* In the *third lesson*, we become familiar with the Cavalieri's principle for geometric figures made out of cubes, where this principle is especially clear. The calculation of volumes of such figures allows us obtain, by geometric means, the value of the sums

$$1 + 2 + \cdots + n \quad \text{and} \quad 1 + 4 + \ldots + n^2$$

This lesson (except for its very end) is intended for 7th grade students.

In the *fourth lesson*, the Cavalieri's principle is used to compute the volume of the cone and the ball. At the end of the lesson, we calculate the surface areas of the disc and the sphere. This lesson is intended for 8-9th grade students, but can be used as additional material for 10-11th grade students in the space geometry course.

Our approach to area and volume is not formal, but at some stage it is expedient to learn about the axiomatic definition of area. This is done in Supplement A.

Different versions of problems for the math circle (together with additional problems — which include problems of the type studied in the lessons, as well as difficult original problems) appear in Supplement B.

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Lesson 1

Scale and Volume

It is worth starting the acquaintance with the notion of dimension (in the sense of length, area, volume) by considering its "checkered" (i.e., discrete) version, in which any consideration can be verified by direct counting.

You can begin with the following well-known problem, which is often solved incorrectly.

Problem 1.1. After seven washes, the length, width, and height of a bar of soap has been halved. How many more washes will the remaining bar take? (For each wash, the same amount of soap is used.)

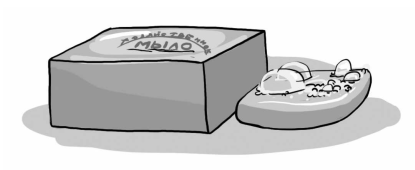


Fig. 1.1 a

The answer given to this question is often the following: "the soap will last for 7 more washes". But if we assume that this is so, then the whole bar of soap is good for 14 washes, i.e., for as many washes as two small pieces of soap (of the size obtained after 7 washes). Let us think: How many such small pieces must be put together to get the initial bar of soap?

If since it is hard to visualize the three-dimensional situation considered in this case, it is expedient to first look at the case of a *flat* square soap; there we immediately see that the big bar of soap consists of four small ones.

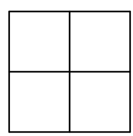


Fig. 1.1 b

In fact, as we shall see in the next problems, the big bar of soap actually consists of eight small bars; accordingly, the first 7 washes consume 7 small bars, so that the remaining small bar is good for only one wash.

If the lesson starts with this problem, then it is not necessary to immediately present its solution in detail. You need only explain that there is something wrong with the naive erroneous answer indicated above.

Problem 1.2. A square of side (a) 3 cm; (b) 1 m was cut into squares of side 1 cm. How many squares were obtained?
A cube of side (c) 3 cm; (d) 1 m was cut into cubes of side 1 cm. How many small cubes were obtained?

It is always difficult to calculate something directly from a three-dimensional image. Usually one has to reduce the problem to a specially constructed flat problem. One way to do this is to consider the 3D-image layer by layer ("floorwise").

Solution. (c) A $3 \times 3 \times 3$ cube consists of three identical layers. Each of these layers is a square of size 3×3, which, as we found out in the previous problem, consists of $3 \times 3 = 9$ square cells. So there are $3 \times 9 = 27$ small cubes.

d) Similarly, we find that a cube of side 1 m consists of $100^3 = 1\,000\,000$ cubes of side 1 cm (this explains why one cubic metre is not one hundred, but one million cubic centimetres).

In general, cutting a cube of side 1 m, for example, into fairly small cubes, one can construct an arbitrarily high tower (also see Problem 2.11).

Problem 1.3. A loader in a warehouse can lift a package of $3\times3\times3$ one-litre cartons of milk. Will three loaders be able to lift a package of $9 \times 9 \times 9$ cartons?

Solution. If we simply count the weight of a large package consisting of $9\cdot9\cdot9 = 729$ cartons, which is approximately 729 kg, it will become clear that the three loaders cannot lift it.

In any case, it is worth figuring out how many small packages the big one consists of. But it is easy to see that we have actually solved this problem, see above (with cubes instead of milk cartons), and the answer will be the same: the large package is 27-times heavier than the small one.

Problem 1.4. A children's inflatable pool is 30 cm deep, and its bottom is a square of side 1 m. What is the weight of such a pool filled with water?

Solution. Let us recall that 1 litre, that is, a cubic decimetre, of water weighs 1 kg. Therefore, the weight of a pool with water in kilograms is equal to its volume in dm^3. Accordingly, the volume of our water-filled pool is $3 \cdot 10^2 = 300$ dm^3 and its weight is 300 kg.

Problem 1.5. Alex and John each built a tower of cubes (see Figure 1.5). Both towers have a square base and consist of the same number of cubes.

(a) The side of the base of John's tower is four times longer than Alex's. How many times higher is Alex's tower than John's?

(b) Alex's tower is four times higher than John's. How many times longer is the side of the base of John's tower than that of Alex's tower?

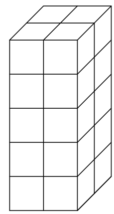


Fig. 1.5

Answer: (a) 16 times; (b) twice.

The central question of the lesson is how the volumes and areas of arbitrary shapes change when their linear dimensions are multiplied by a positive integer k (we have already encountered this question in its simplest form in Problem 1.3).

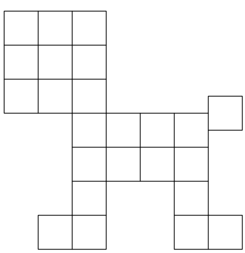


Fig. 1.6 a

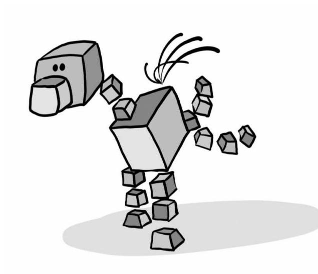


Fig. 1.6 b

Problem 1.6. (a) Alex put together a picture consisting of squares of side 2 cm (see Figure 1.6 (a), and John, a similar picture of squares of side 4 cm. How many times is the area of Alex's picture smaller than John's?

(b) A cubic figure is made up of several wooden cubes (see Fig. 1.6 (b). How will its mass change if the size of each cube is doubled?

It is also worth finding out what the answer will be if the size is increased k times rather than doubled. Note that the answer is completely independent of the shape of the cubic figure. From this we can conclude that the next problem has the same answer.

Problem 1.7. How will the mass of an elephant change if all of its dimensions are doubled? (Of course, you are not allowed to assume that an elephant has the shape of a parallelepiped.) How will the area of the elephant in the snapshot change?

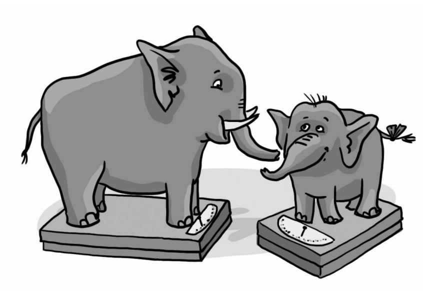


Fig. 1.7 a

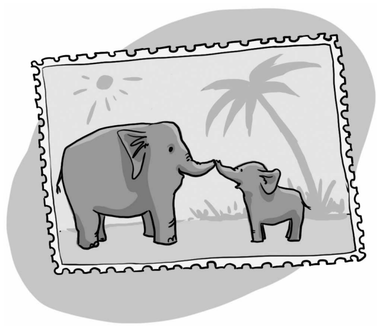


Fig. 1.7 b

Answer: 8 times; 4 times.

Solution. To associate this problem with the previous one, we can first imagine that the elephant is composed of small cubes ("pixels"). Now, as the cubes become very small, the pixel elephant becomes indistinguishable from the real one ...

Of course, no formal proof in this problem is required; it suffices to understand what the answer will be.

In fact, as the size of the elephant increases, the elephant's volume, and hence its mass, will increase as the cube of its linear dimensions, while the cross-sectional area of each leg, and, consequently, the strength of its bones will increase only as the square of linear dimensions. This means that, as the elephant's size increases, its mass will grow significantly faster than the strength of its legs, the enlarged elephant will no longer be able to stand on its feet.

The same effect can be seen in the following simple discrete model: if a big cube is composed of small cubes, then the load on each small cube will increase in proportion to the size of the big cube, simply because the tower of cubes above it will increase in height. Therefore, at some point, the big cube will collapse under its own weight.

If you know how areas and volumes change under scaling, then it is not difficult to understand (qualitatively) what various formulas for areas and volumes should look like.

Problem 1.8. (a) Denote the area of a circular disk of radius 1 by V_2. What is the area of a disk of radius R?

(b) Denote the volume of a ball of radius 1 by V_3. What is the volume of a ball of radius R?

Answer: (a) V_2R^2; (b) V_3R^3.

The task of calculating the constants V_2 and V_3 is much more subtle. It can be shown (see Lesson 4) that $V_2 = \pi$, $V_3 = \frac{4}{3}\pi$.

Additional problems

Problem 1.9. Which of the pots can be filled with the most amount of liquid, the left one, which is wider, or the right one, which is three times higher, but twice narrower?

Answer: The left pot ($1\frac{1}{3}$ times).

Fig. 1.9

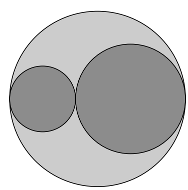


Figure 1.10

Problem 1.10. Two balls of radii 3 and 5 were placed on the left pan of the scales and one ball of radius 8 on the right pan. Which of the pans will outweigh the other? (All balls are made of the same material.)

A typical answer to such a question is "None, because $3 + 5 = 8$".

Solution. This answer can be refuted convincingly by a visually clear geometric argument: the two small balls put side by side will fit into the big one; hence, their total volume must be less than that of the big ball.

Problem 1.11.* Two round coins were put on the left pan of the scales and another one was put on the right pan, and the scale was in balance. Which of the pans will outweigh the other if each of the coins is replaced by a ball of the same radius? (All the balls and coins are made of the same material, and all the coins have the same thickness.)[1]

Solution. Denote the radii of the coins by R_1, R_2, and R_3. Since, at first, the scales were in equilibrium, we have $V_2R_1^2 + V_2R_2^2 = V_2R_3^2$, that is, $(R_1^2 + R_2^2 = R_3^2)$. Similarly, to determine what happened to the scales after the coins were replaced by balls, we must compare $R_1^3 + R_2^3$ to R_3^3. But, considering the equality above, we see that the right-hand side is now multiplied by the larger radius R_3, while the two summands on the left-hand side by the smaller radii R_1 and R_2:

$$R_1^3 + R_2^3 = R_1^2 \cdot R_1 + R_2^2 \cdot R_2 < R_1^2 \cdot R_3 + R_2^2 \cdot R_3 = (R_1^2 + R_2^2) \cdot R_3 = R_3^3.$$

So, the right pan will outweigh the left one.

[1]The author of this problem is G. Galperin; it was presented at the Lomonosov Tournament in 2009.

Lesson 2

Surface Area

Problem 2.1. What is the surface area of a cube of side (a) 10 cm; (b) 12 cm?

Answer: (a) $6 \cdot 10^2 = 600$ (cm^2); (b) $6 \cdot 12^2 = 864$ (cm^2).

In Problem 5 of the previous lesson, we already saw that if a cube is cut into sufficiently small cubic bricks, then an arbitrarily high tower can be built from these bricks. The following question arises: How large is the surface area of this tower? It appears that the tower is narrow and the surface area must be small. Let us check this.

Problem 2.2. A cube of side 1 m is cut into cubic bricks of side 1 cm from which a tower with base of side 1 is built. What is the area of the tower's surface? Is it smaller or greater than the surface area of the initial cube? How many times?

Answer: $100^3 \cdot 4 + 2$ (cm^2) ≈ 400 (m^2); this is about $\frac{400}{6} \approx 67$ times greater than the surface area of the initial cube.

Problem 2.3. A cube of side 12 built from small cubic bricks of side 1 is painted white. How many bricks have no faces painted; one face painted; two faces painted; three faces painted?

Solution. The bricks without colored faces form a cube of size $10 \times 10 \times 10$ — as we already know, it consists of $10^3 = 1000$ bricks. The bricks with three colored faces are the corner ones, and there are as many as there are vertices in the cube, that is, eight. All bricks with one face painted have their painted face strictly inside the faces of the cube, and their number is $6 \cdot 10^2 = 600$ (cf. Problem 2.1 (a)).

Finally, the bricks with two faces painted are those on edges of the initial cube except the corner ones, and their number is $12 \cdot 10 = 120$.

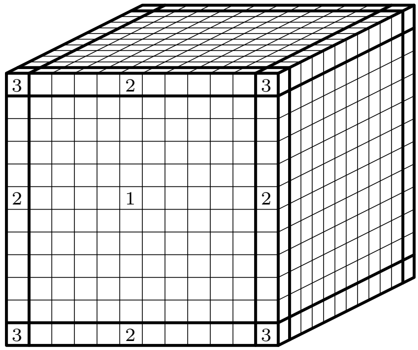


Fig. 2.3

Adding all these numbers together, we obtain $12^3 = 10^3 + 6 \times 10^2 + 12 \cdot 10 + 8$; in general, increasing the side of the cube from a to $a+b$ and performing a similar calculation, we can prove the well-known identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, which is a special case of *Newton's binomial formula*. The general binomial formula can be proved by a similar argument.

Problem 2.4. (a) What fraction of the area of a square of size 12×12 is taken up by the boundary cells?

(b) What fraction of the volume of a cube of size $12 \times 12 \times 12$ is taken up by the boundary bricks?

Answer: (a) 31%; (b) 42%.

Solution. (b) At first glance, it suffices to repeat the solution of Problem 2.1 (a): the cube has six faces, each of which borders on $12 \cdot 12 = 144$ bricks; hence there must be $6 \cdot 144 = 864$ bricks in all.

But this calculation yields a wrong answer, because some bricks border on more than one face and, accordingly, are counted several times. It is easy, however, to rectify this by using the argument of the preceding problem: we are interested in the bricks which have at least one painted face; their number is $600 + 120 + 8 = 728$, and they take up $728/1728 \approx 42\%$.

There is an easier solution of the problem: it consists in counting interior, rather than boundary, bricks. The number of interior bricks is 10^3, which is $\frac{10^3}{12^3} = \left(\frac{5}{6}\right)^3 \approx 58\%$. Accordingly, the boundary bricks take up $100\% - 58\% \approx 42\%$.

Note that the fraction of boundary bricks rapidly increases with dimension (to make this particularly evident, we have added the answer "17%" to the same question about a segment of length 12 composed of segments of length 1). Later on we shall observe the same effect in the continuous situation.

Problem 2.5. A food market sells two varieties of watermelons of the same diameter. A watermelon of the first variety costs 100 rubles, but has a very thin peel. A watermelon of the second variety costs 70 rubles, but its peel (which must be thrown away!) takes up 20% of the diameter. Which variety of watermelons is it more profitable to buy?

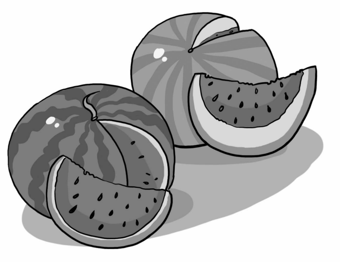
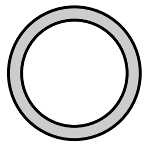
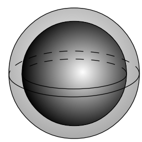


Fig. 2.5

After an exchange of informal arguments, the teacher can take a poll on which variety of watermelons are preferred. The voices will split, but most likely, the majority will vote for buying seventy-ruble watermelons, because they seem to contain 20% less pulp, while their price is 30% lower.

Solution. To solve this problem, it is not necessary to know the exact formula for the volume of a ball; it suffices to understand how this volume changes with the linear size of the ball (see Problems 1.7 and 1.8). The *radius* of the pulp in a watermelon of the second variety is 0.8 times of that in a watermelon of the first variety. But to find how many times smaller is its volume, it is required to take the cube of side 0.8: $0.8 \cdot 0.8 \cdot 0.8 = 0.64 \cdot 0.8 = 0.512$, so that *almost half* of a watermelon of the second variety is occupied by peel! Thus, of course, we should buy the first watermelon.

It is instructive to find out how the fraction of the volume of peel taking up 20% of the radius depends on the dimensionality of the "watermelon". It is seen from the table that this fraction grows fairly rapidly with dimension.

Dimension	Peel fraction	Picture
1	$1 - 0.8 \;= 0.2$	
2	$1 - 0.8^2 = 0.36$	
3	$1 - 0.8^3 \approx 0.49$	
4	$1 - 0.8^4 \approx 0.59$	—

Problem 2.6. The width of a flat copper ring increases 1.5 times when the ring is heated. How does the area of the hole change?

Answer: Increases $1.5^2 = 2.25$ times.

Problem 2.7. The length of the equator of a globe equals 1 m. (a) What is the scale of the globe? (b) What is the area of Russia on this globe? (The length of the Earth's equator equals 40 000 km; the area of Russia is about 17 000 000 km^2.)

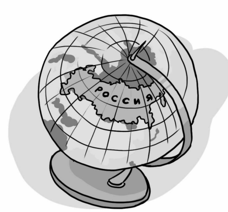


Fig. 2.7

Answer: (a) 1 : 40 000 000; (b) approximately 106 (cm^2).

It is important to understand that expanding by a factor of k increases the area of all figures (not only planar ones) by a factor of k^2.

Problem 2.8. The Earth was hooped along the equator. Then the hoop was lengthened by 1 m (so that the arising gap was the same everywhere). Can a cat slip under the hoop?

Answer: Surprising as it may seem, it can (the clearance is about 16 cm).

To figure out this problem, it is useful to begin by solving its discrete version.

Fig. 2.8

Problem 2.9. A paper belt is put on a cube of size (a) $3 \times 3 \times 3$; (b) $100 \times 100 \times 100$ as tightly as possible. What clearance will result from lengthening the belt by 8 (so that the belt will remain square)?

Solution. It is easier to solve the inverse problem: How does the length of the belt change when the gap increases? When the "radius" of the belt increases by 1, each of its sides increases by 2 and its length, by $4 \cdot 2 = 8$. Therefore, conversely, when the length of the belt increases by 8, a gap of size 1 appears.

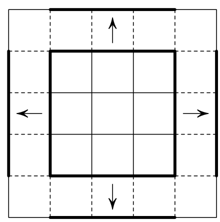


Fig. 2.9

Now we can solve the original problem. We again begin with the inverse problem: How does the length of the hoop change when its *radius* increases from R to $R + \delta$? It is easy to see that the length increases by $2\pi(R+\delta) - 2\pi R = 2\pi\delta$. Therefore, increasing the *length* of the hoop by 1 m results in a clearance of $\delta = \dfrac{1}{2\pi} \approx 0.16$ cm.

Note that, in both problems, the answer does not depend at all on the initial size (of the cube or of the Earth). This is a manifestation of *linearity* of the problem (cf. Problem 2.10.)

Additional Problems

Problem 2.10. A balloon (having the shape of an ideal ball) was blown up so that its area increased by 9%. What happened to its radius?

Answer: The radius increased $\sqrt{1.09} \approx 1.044$ times, that is, by approximately 4%.

Note that, unlike in Problem 2.8, the increase in the radius in centimetres cannot be found from the increase in the area in square metres.

Problem 2.11. Is it possible to cut out several discs from a square of side 10 cm and put them in line so as to obtain a chain longer than one kilometre?

Solving this problem, pupils often try to first cut out the largest disc, that is, the disc inscribed in the square.

But after that, difficulties arise: of course, one can cut out four discs from the remaining corners, but they are very small (what is their diameter?) and continue to cut out the biggest possible (although rapidly decreasing) discs. At this point, some students will say: "Since we can cut out discs as long as we like, we can make the sum of their diameters arbitrarily large". However, one should remember the example of an infinitely decreasing geometric progression (e.g., in which each term is half as long as the preceding one) – this shows that such an argument is insufficient for solving the problem.

Solution. First, dividing the sides of the initial 10×10 cm square into halves, we split the initial square into four squares with a halved side of 5 cm, and cut out a disc from each of them. We obtain four discs, each of diameter 5 cm, so that the sum of their diameters (20 cm) is twice as large as the side of the square. What if we divide the sides into three equal parts? Of course, we obtain nine squares of side 10/3 cm each. The sum of the diameters of their inscribed circles is $9 \cdot (10/3) = 30$ cm. Similarly, dividing the sides into ten parts, we obtain 100 circles of diameter 1 cm, whose diameters sum up to exactly one metre. In general, dividing the sides into n parts, we obtain n^2 squares of side $10/n$, and the sum of the diameters of inscribed circles equals $n^2 \cdot (10/n) = 10n$, so that increasing n, we can make it arbitrarily large.

Parts	Number of squares	Disc diameter	Sum of diameters
1	1	10 cm	10 cm
2	4	5 cm	20 cm
3	9	10/3 cm	30 cm
10	100	1 cm	100 cm
10000	10^8	0.01 mm	1 km
n	n^2	$10/n$ cm	$10n$ cm

The essence of the solution is that, when the square is cut into small squares, the side of each small square decreases linearly, while their number grows quadratically. Accordingly, the sum of diameters of the circles inscribed in the small squares depends linearly on their number. Therefore, increasing n, we can make this sum greater than any given number.

However, to fulfil the requirements in the problem (to built a chain of length at least 1 km), the number of parts will be $n = 10^4$ and we will have to cut up the initial square into 10^8 small squares of side 10^{-3}cm, which, of course, is hard to do in practice.

Lesson 3

Areas and Sums

As we already saw in Problem 1.2, sometimes it is useful to examine figures layerwise, since this allows us to represent volumes and areas of checkered figures as sums. This method works in both directions: sometimes, looking at such a sum, we can understand something about the area (e.g., as in Problem 3.1 (a)), and sometimes, the other way around—representing a sum as an area or a volume, we can calculate it from geometric considerations (as in Problem 3.4 or 3.10*).

It is useful to model the figures discussed in the problems of this lesson (e.g., by glueing together toy bricks). Armed with such models, you can give even Problem 3.10* to kids of any age (in the setting "Build a parallelepiped (brick) from several pyramids").

Problem 3.1. (a) Which of the figures in Figures 3.1a and 3.1b contains more squares? (b) Count these squares.

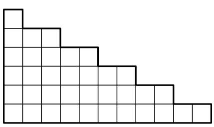


Fig. 3.1a

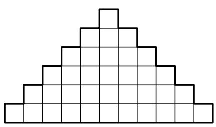


Fig. 3.1b

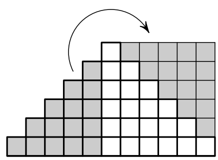


Fig. 3.1c

Answer: (a) Equally many; (b) 36.

Solution. (a) The respective "rows" of the figures in the figures contain equally many squares (1, 3, 5, and so on). Hence the total numbers of squares are equal as well.

(b) The problem can be solved by simply summing the numbers of squares in layers: $1 + 3 + 5 + 7 + 9 + 11 = 36$. But it can also be solved more geometrically: cutting the second figure into two parts and assembling the parts into a square (see Fig. 3.1c), we see that the figure consists of $6^2 = 36$ cells.

The latter solution gives a geometric proof of the identity $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

Certainly, if such an identity is *already written*, it is possible (and even easier) to prove it by induction. But geometric summation is a method for not only proving, but also finding, similar identities; see also Problem 3.10*.

Problem 3.2. A triangle lies in a rectangular box, so that one of its sides coincides with the bottom edge of the box and the remaining vertex lies on the opposite edge (see Fig. 3.2a). What fraction of the area of the box does the triangle occupy?

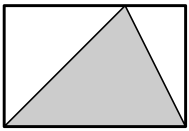


Fig. 3.2a

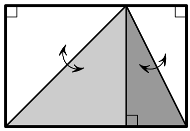


Fig. 3.2b

Answer: One half.

Solution. Let us mentally split the box into two parts (see Fig. 3.2b). Exactly half of each of them is taken up by the triangle. Therefore, in the whole box, exactly half the area is taken up by the triangle.

This argument *proves* that the area of a triangle equals half the product of the base by the altitude[2]. In general, this problem can serve as the beginning of a conversation about proving formulas for the areas of figures (triangle, parallelogram, trapezoid) and, if desired, about the definition of area (any calculation of an area is based on cutting figures into triangles and assembling rectangles from triangles). But the main topic of this lesson is different.

Problem 3.3. (a) Another triangle was put sideways into a box (see Fig. 3.3a). Is the fraction of the box area occupied by it greater than, smaller than, or equal to that in the previous problem? (b)* Does a triangle of area 10 fit in a box of area 19?

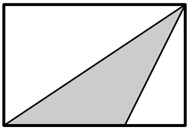


Fig. 3.3a

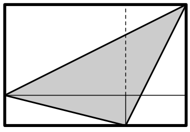


Fig. 3.3b

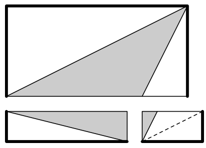


Fig. 3.3c

[2]Provided that the base angles are not obtuse.

Answer: (a) Smaller; (b) no.

Solution. (a) The longest side of the triangle divides the box in halves, but the triangle itself occupies only part of one of the halves.

(b)* A triangle cannot occupy more than half the area of a rectangular box. To prove this, consider the possible locations of the vertices (we may assume that all of them lie on the side of the rectangle, since otherwise we can reduce the size of the box): either one of the edges of the box contains two vertices of the triangle (this case has already been handled in the preceding problem and in part (a) of this one), or the vertices lie on three sides, that is, to be more precise, on two sides and at a corner (see Fig.3.3b), in which case the box can be split into three parts,in each of which the triangle occupies at most half of the area (see Fig. 3.3c).

Problem 3.4. The area of an isosceles right triangle is half the area of a square whose side is equal to the leg of this triangle. What is the area of a "pixel" (composed of unit squares) isosceles right triangle with leg, say, 20 (see Fig. 3.4a)?

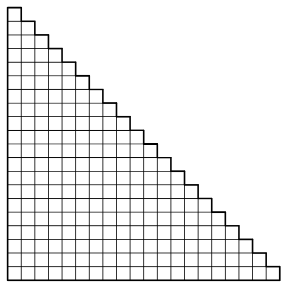


Fig. 3.4a

Answer: $\dfrac{20 \cdot 21}{2} = 210.$

Hint. Try to put such a pixel triangle next to another one (of the same size) in a square box of side 20. How many cells will not fit?

Solution. *First solution.* It is easy to make up a rectangle of size 20×21 from two such pixel triangles (see Fig. 3.4b). Thus, the area of the triangle is half the area of this rectangle and is equal to $(20 \cdot 21)/2 = 210$.

Second solution. First, let us look carefully at the true (not pixel) triangle. It occupies precisely half the box area because of symmetry: the unoccupied part is perfectly symmetric to the occupied one.

Let us try to repeat this argument for the pixel triangle. The reflection of the given triangle across the diagonal intersects the original triangle (see Fig. 3.4c). But the cells in the intersection are easy to count: all of them lie on the diagonal of the square, and their number is exactly 20. Thus, if the required area equals S, then the area of the square equals $20^2 = 2S - 20$, whence we obtain $S = \dfrac{20^2 + 20}{2} = 210$.

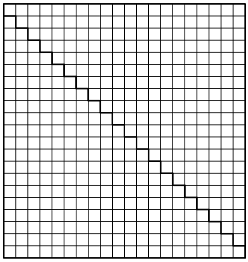


Fig. 3.4b

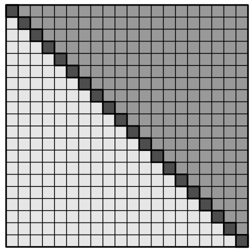


Fig. 3.4c

The second solution may seem to be more complicated, but it is based on a more powerful idea, so that it extends more readily to other sums (see, e.g., Problem 3.10*; the second solution can well be postponed until its discussion).

Definition. The area of a right isoceles pixel triangle with leg n (that is, the number $1 + 2 + 3 + \ldots + n$) is called the nth *triangular number* and denoted by T_n.

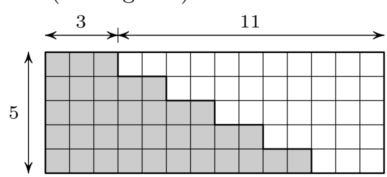
Problem 3.5. Find the hundredth triangular number.

Hint. One can look for it geometrically by using the preceding problem.

Answer: $\dfrac{101 \cdot 100}{2} = 5050$; in general, $T_n = \dfrac{n(n+1)}{2}$.

Clearly, when we reduce the size of the pixels, the pixel triangle tends to the true one. It is instructive to check that its area tends to the area of the true triangle as well.

In the same way, we can find the sum of any arithmetic progression: first, we represent it as the area of a right pixel trapezoid and then make up a rectangle from two equal trapezoids (see Fig. 3.5).



$2\cdot(3+5+\ldots+11)=(3+11)\cdot 5$

Fig. 3.5

As we see, the height of the resulting rectangle is equal to the altitude (that is, to the number of "terms") of the trapezoid, and its width is equal to the sum of the lengths of its bases (that is, the sum of the first and the last term).

Problem 3.6. Find the sum of all two-digit numbers divisible by 7.

Answer: $\dfrac{(14+98)\cdot 13}{2} = 728.$

Solution. We are asked to find the sum $14+21+\ldots+91+98$. This is the sum of an arithmetic progression consisting of $(98-14)/7+1 = 13$ terms, which, as explained above, can be calculated geometrically.

Problem 3.7. Find the sum of two consecutive triangular numbers.

Hint. Two pixel triangles of sizes T_n and T_{n-1} can be assembled into an $n \times n$ square.

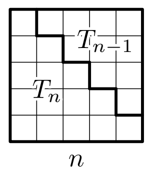


Fig. 3.7

Answer: $T_n + T_{n-1} = n^2$.

Problem 3.8. Which of the figures shown in Figures 3.8a and 3.8b contains more cubes?

Hint. Cut the picture into layers and apply Problem 3.1 to each of them.

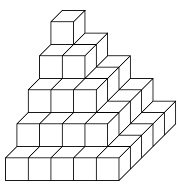


Fig. 3.8a

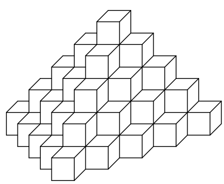


Fig. 3.8b

Problem 3.9. What part of a cubic box is occupied by the (irregular) quadrangular pyramid shown in Fig. 3.9a?

Certainly, in this problem the known formula for the volume of a pyramid cannot be used (in return, as we shall see in the next lesson, the volume formula can be derived from this problem).

Hint. Fill the whole box with several such pyramids.

Answer: 1/3.

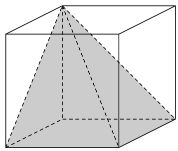


Fig. 3.9a

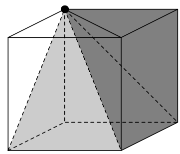


Fig. 3.9b

Solution. Three such pyramids can be assembled into a cube (see Fig. 3.9b). This can be explained as follows. Choose one of the vertices of the cube and consider the three faces not containing it. Let us construct the three pyramids whose apexes coincide with the chosen vertex and whose bases coincide with the chosen faces. These pyramids are shown in different shadings.

The same partition can be written in coordinates: the cube $0 \leqslant x_1, x_2, x_3 \leqslant 1$ is split into the parts $P_i = \{(x_1, x_2, x_3) \mid \max(x_1, x_2, x_3) = x_i\}$. Note that such a partition generalizes directly to a (hyper)cube of any dimension.

Problem 3.10*. (a) In the corner of a room a pyramid of altitude n is stacked (see Fig. 3.8a). How many cubes does it contain?

(b) Calculate the sum $1^2 + \ldots + n^2$.

Hint. Try to make up a figure of already known volume from several such pyramids (recall Problem 3.4 for inspiration).

Answer: $\dfrac{n(n+1)(2n+1)}{6}$.

Solution. One of the solutions of this problem reduces to assembling the six pyramids of the preceding problem into a parallelepiped of size $n \times (n+1) \times (2n+1)$, in the spirit of the first solution of Problem 3.4 (Figure 3.10a shows the three successive steps of assembling three such pyramids into half of such a parallelepiped; see also [5]). The drawback of this approach is that the picture is complicated by the "teeth" of the pyramids, so that it is even unclear whether the sum of third powers can be found in this way.

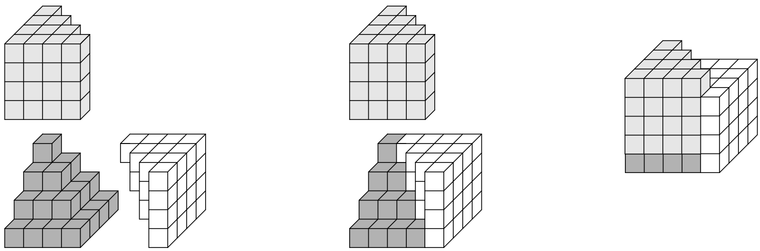


Fig. 3.10a

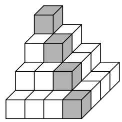


Fig. 3.10b

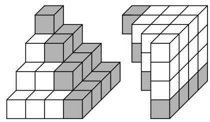


Fig. 3.10c

A more satisfactory method is to keep what is necessary (namely, the partition of the cube into three equal parts) and throw out the inessential: we will not insist on the disjointness of the parts. Instead, we partition the cube into three *intersecting* pyramids and then take into account their intersections (in the spirit of the second solution of Problem 3.4).

Thus, we have split the cube of size $n \times n \times n$ into three pyramids. The common part of all three pyramids is the diagonal (see Fig. 3.10b), i.e., it consists of precisely n cubes. The pairwise intersections of pyramids are pixel triangles (see Fig. 3.10c) containing $1 + \ldots + n$ cubes each.

It remains to apply the inclusion–exclusion principle: denoting $1^k + \ldots + n^k$ by $S_k(n)$, we obtain $n^3 = 3S_2(n) - 3S_1(n) + n$, and recalling that

$$S_1(n) = \frac{n(n+1)}{2},$$

we find

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Additional Problems

Problem 3.11.* Prove the addition theorem for triangular numbers: $T_{n+m} = T_n + T_m + nm$.

Hint. Build an isosceles right triangle with leg $n + m$ from two triangles with legs n and m and a rectangle with nm cell (see Fig. 3.11).

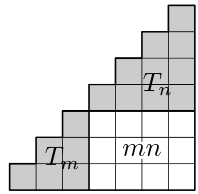


Fig. 3.11c

Problem 3.12*. Acting in the spirit of the second solution of Problem 3.10*, try to successively find the sums of third and fourth powers ($S_3(n)$ and $S_4(n)$).

Answer: $S_3(n) = \left(\frac{n(n+1)}{2}\right)^2$;

$$S_4(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

A geometric proof of the identity $S_3(n) = S_1(n)^2$ can be found in the paper [1].

Note that $S_4(n)$ does not decompose into linear factors. Thus, it is hardly possible to find this sum by "assembling a (hyper)parallelepiped from (hyper)pyramids.

Problem 3.13*. Find the nth *pyramidal number*, that is, the sum of $T_1 + T_2 + \ldots + T_n$ consecutive triangular numbers.

Hint. Construct a parallelepiped from pyramids similar to those in Problem 4.1.

Answer: $\frac{n(n+1)(n+2)}{6}$.

Lesson 4

Cavalieri's Principle

Problem 4.1. Which of the pyramids shown in the figures has more cubes?

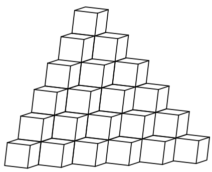


Fig. 4.1a

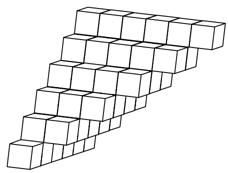


Fig. 4.1b

Solution. Let us look at the vertical layers. Each of them is a "triangle" of cubes, and the corresponding layers of pyramids coincide (up to translation). Therefore, the pyramids have the same number of cubes.

The solution of this problem is based on the idea of considering 3D pictures layer by layer, which has played the key role in the preceding lesson. The rest of the lesson is devoted to applying the same idea, but in the continuous, rather than discrete, context.

Problem 4.2.* Prove that

$$T_1 + T_2 + \ldots + T_n = n \cdot 1 + (n-1) \cdot 2 + (n-2) \cdot 3 + \ldots + 1 \cdot n.$$

Hint. Cut the pictures to the previous problem into horizontal layers.

Given two solids in space, suppose that all planes parallel to a fixed one are drawn. *Cavalieri's principle* asserts that if the area of the section of the first solid by each plane is equal to the area of the section of the second solid by the same plane, then the volumes of the solids are equal.

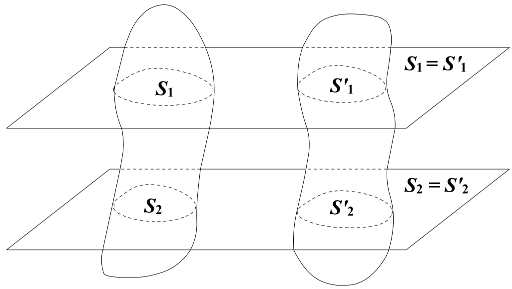


Fig. 4.2

This statement can be generalized: if the respective areas of sections of two solids differ by a factor of k, then the volumes of these solids differ by the same factor.

Problem 4.3. Formulate an analogue of Cavalieri's principle for planar figures.

Solution. Given two figures in the plane, suppose that all straight lines parallel to a fixed one are drawn. If each of the lines intersects the figures in equal segments, then the areas of the figures are equal (see Fig. 4.3).

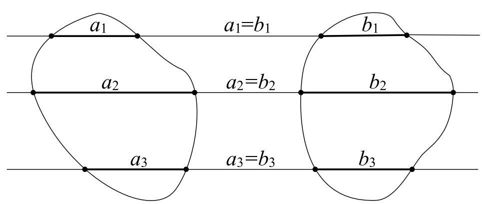


Fig. 4.3

Problem 4.4. Prove Cavalieri's principle (a) for trapezoids with bases parallel to the sections; (b) for convex polygons in the plane.

Solution. (a) The required assertion follows directly from the area formula for a trapezoid.

(b) Suppose that the respective sections of polygons M and N by straight lines parallel to a line l are equal. Let us draw the line parallel to l through each of the vertices of M and N (see Fig. 4.4). These lines splits each polygon into trapezoids (in the generalized sense: some of these trapezoids may be parallelograms or even triangles). The areas of the respective trapezoids are equal by virtue of part (a) of the problem. Therefore, so are the areas of the polygons.

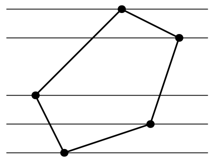


Fig. 4.4

A similar argument shows that if the lengths of respective sections of two polygons by straight lines parallel to a given one differ by a factor of k, then the areas of these polygons differ by the same factor.

Problem 4.5. A solid is expanded by a factor of k in one direction. What happens to its volume?

Answer: It is multiplied by k.

If you desire, you need not derive this fact from Cavalieri's principle and consider it as being obvious.

Solution. The answer can be formally derived from Cavalieri's principle as follows. Choose a plane α parallel to the direction of expansion. Then each section of the new solid by a planes parallel to α is the expanded (by a factor of k) section of the old one by the same plane. Therefore, the areas of these sections differ by a factor of k (see the comment concerning the preceding problem). Thus, the volumes of the old and new solid differ by a factor of k as well.

This allows us to rigorously derive the answer to Problem 1.7, which we have taken on faith: to k-times enlarge an elephant in all dimensions, we expand in each of the three coordinate directions in turn; each expansion increases the volume k times, so that in all the volume will increase k^3 times.

Definition. In this lesson, by a *cone* we mean a body consisting of a flat figure ('the base of the cone') together with all straight line segments joining it with a fixed point ('the vertex, or apex, of the cone') outside the plane of the base.

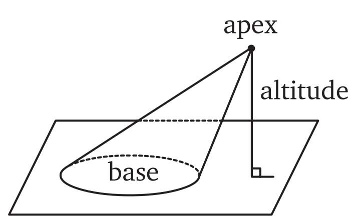


Fig. 4.5a

For example, the usual ('right circular') cone is the cone whose base is a disc (and the vertex is exactly above the centre of this disc), its lateral surface is a cone over a circle, and a pyramid is a cone over a polygon.

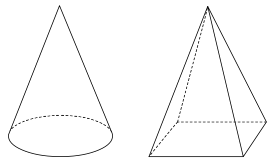


Fig. 4.5b

Problem 4.6. Suppose that a cone has a base of area S and its height equals h. Find the area of the section of this cone by a plane parallel to the base at a distance of x from the vertex.

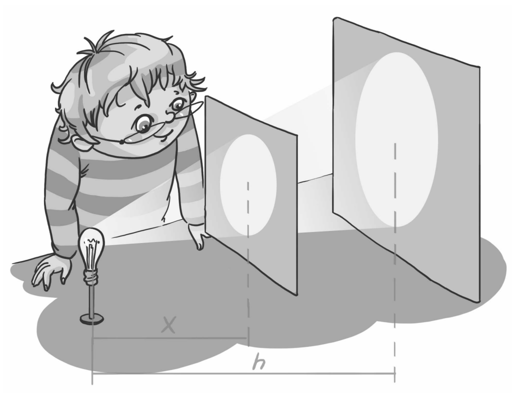


Fig. 4.6

Solution. Note that this section is the base reduced by a factor of h/x. Therefore, the required area equals $(x/h)^2 S$.

Problem 4.7. Prove that the volume of a cone depends only on its altitude and on the area of its base (and does not depend on the shape of the base).

Problem 4.8. (a) Given that the volume of a cone of altitude 1 with base of area 1 equals c, calculate the volume of a cone of altitude h with base of area S. (b) What is c?

Hint. b) It is convenient to consider a cone which is a pyramid with square base.

Answer: (a) chS; b) $c = \frac{1}{3}$. Thus, the volume of the cone is $\frac{1}{3}hS$.

Solution. (b) See Problem 3.9.

Problem 4.9. Find the area of the section of a ball of radius R by a plane at a distance of x from the centre (see Fig. 4.9).

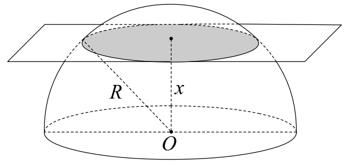


Fig. 4.9

Answer: $\pi(R^2 - x^2)$.

Problem 4.10. Find the volume of a ball of radius R.

Hint. Using Cavalieri's principle, represent this volume as the difference of the volumes of two bodies.

Solution. Consider the hemisphere (see Fig. 4.10). As is seen from the previous problem, the area of each of its sections is the difference of two areas, πR^2 and πx^2, that is, the areas of the corresponding sections of the cylinder and the cone. It follows (from Cavalieri's principle) that the volume of the hemisphere is the difference between the volume of the cylinder (of altitude R with base of radius R) and the cone (with the same altitude and base). Thus, the volume of the ball of radius R is

$$2(\pi R^3 - \frac{1}{3}\pi R^3) = \frac{4}{3}\pi R^3.$$

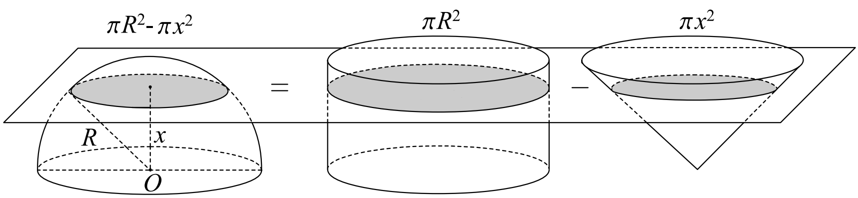


Fig. 4.10

This calculation of the volume of a ball is due to Archimedes. In our times, the volume of a ball is usually computed by integration, although the calculation is essentially the same.

Problem 4.11. (a) On a square lined paper, a polygon M whose vertices are at grid points and sides do not go along grid line is sketched. Prove that the sum of the lengths of vertical segments of the grid lines inside M is equal to the sum of lengths of horizontal segments of the grid lines inside M.

(b) What is a generalisation of this statement to polygons whose sides are allowed to go along grid lines?

Hint. Each of the sums is equal to the area of the polygon.

Compare this problem with Problem 4.4.

Solution. (a) Let us cut M along the horizontal grid lines. We obtain several triangles, trapezoids, and possibly parallelograms. The area of each of these figures is equal to half the sum of the two horizontal segments of the grid lines that bound it (if a figure is bounded by one segment, then the second number is assumed to be zero). Summing these areas and noticing that each segment occurs precisely twice in the sum (it bounds some figure from above and another figure from below), we obtain the assertion of the hint.

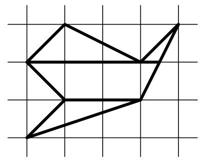


Fig. 4.11

(b) It is seen from the solution of part (a) that the sides going along the grid lines must be assigned the weight 1/2. As a result, we obtain the following assertion.

Given an arbitrary polygon on square lined paper with vertices at grid points, the sum of the horizontal segments of the grid inside the polygon plus half the sum of its horizontal sides is equal to the sum of vertical segments of the grid lines inside the polygon plus half the sum of its vertical sides (and is equal to the area of the polygon).

The statement of part (b) clarifies the solution of part (a): it is easy to understand that if the assertion of the hint holds for two polygons without interior points, then it holds also for their union; it remains to note that any polygon can be cut into small trapezoids and triangle (with sides going along grid lines), for which the required assertion is obvious.

Supplement: Areas of discs and spheres

Having learned how to calculate the volumes of cones and knowing the volume of a ball, we can look into the question of why the area of a disc is πR^2 and the area of a sphere is $4\pi R^2$.

First, we must understand what the question is about: we already saw in Problem 1.8 that the area of a disc is (constant) $\cdot R^2$, and it is easy to calculate the approximate value 3.14 of the constant. Apparently, to be done, it suffices to simply denote this constant by π. The point is that there is yet another definition of the number π: from the same dimensional considerations, the *length* of a circle is (constant) $\cdot R$, and π is defined to be half this constant. What we want to prove is that these two definitions give the same number.

To relate the length of a circle to the area of a disc, we shall think of a disc as of the union of concentric circles. Let us imagine that each of these circles is an elastic string. If we cut our disk along a radius (see Fig. 4.12), then each string will straighten out into a segment. Into what figure will our disc turn? What figure will then be obtained from our disk?

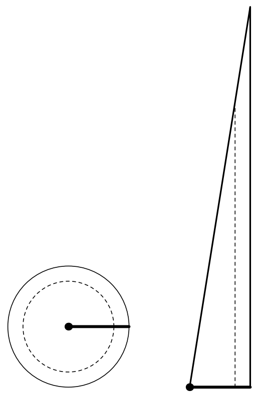


Fig. 4.12

Since the length of the string at a distance x from the center equals $2\pi x$, we obtain a right triangle with legs R and $2\pi R$. The area of this triangle (and hence of the disc of radius R) is $(2\pi R \cdot R)/2 = \pi R^2$, as required.

Such a reduction of the evaluation of the area of a ball to that of the area of a triangle resembles the calculation of the volume of a cone in the last lesson. We can say that we consider the disk as a "cone over a circle" (and accordingly adapt the argument, in the spirit of the calculation of the area of the sphere performed below).

Note also that now it is easy to find the area of any ellipse. Indeed, an ellipse with semi-major axis a and semi-minor axis b is a circle stretched a times in the horizontal direction and b times in the vertical direction. Since the expansion by a factor of k in one direction multiplies the area by k, it follows that the area of the ellipse equals πab.

The area of a sphere can be found in the same way, by considering the ball which it bounds as a cone over it: it suffices to cut the ball into thin wedges with vertices at the centre (see Fig. 4.13).

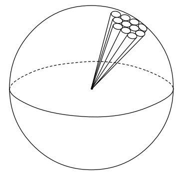


Fig. 4.13

Each wedge can be treated as a cone; therefore, the total volume V of the wedges (that is, the volume of the ball) is $\frac{1}{3}SR$, where S is the sum of the areas of wedge bases (that is, the area of the sphere). Substituting the answer to Problem 4.10, we obtain $SR/3 = 4\pi R^3/3$; thus, $S = 4\pi R^2$.

Notice that the area of the sphere is exactly the product of the circumference of the sphere by its diameter. This is no coincidence: it can be shown that the axial projection of the sphere on a tangent cylinder preserves area—this fact is well known in cartography (see Fig. 4.14).

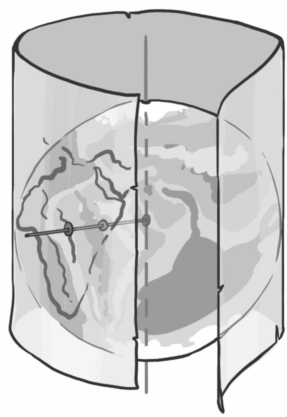


Fig. 4.14

Problem 4.12*. Prove that the ratio of the volume of the ball by that of a polyhedron circumscribed about it is equal to the ratio of the surface area of the ball by that of the polyhedron.

Hint. The ratio of volume to surface area is equal to $R/3$ for both the ball and a circumscribed polyhedron.

Supplement A

Definition of area and volume

The calculation of the area of any figure is based on cutting the figure into pieces and summing the areas of these pieces (see, e.g., the proof of the area formula for a triangle in Problem 3.2). The axiomatic definition of area is based on the same idea.

Area can be defined as a function S on the set of (planar) polygons with the following natural properties ("area axioms"):

(1) S is preserved by motions;

(2) S is strongly additive: if polygons T and T' have no common interior points, then $S(T \cup T') = S(T) + S(T')$;

(3) $S(\text{rectangle of sides } a, b) = ab$.

It can be proved (by cutting any polygon into triangles) that these properties determine the function S uniquely.

If we do not want to limit the definition of area to polygons, then we must add the axiom of area nonnegativity, which makes it possible to calculate the area of a figure by approximating it from below and above by areas of polygons contained inside the figure and containing it. It is also necessary to choose a class of figures for which area is defined, because there does not exist any function with the properties listed above which is defined for all figures, that is, for arbitrary sets of points (see, for instance, the booklet [4]); although, for figures bounded by reasonable curves, no problems of this kind arise.

We have seen that, for calculating volume, Cavalieri's principle is also needed. This is reflected in the definition of volume.

Volume can be defined as a function V on the set of polyhedra which satisfies the following axioms:

(1) V is preserved by motions;

(1′) V satisfies Cavalieri's principle;

(2) if the interiors of polyhedra P and P' do not intersect, then $V(P \cup P') = V(P) + V(P')$;

(3) V(rectangular cuboid with sides a, b, c) $= abc$.

Note that the counterpart of axiom 1′ for polygons follows from axioms 1 and 2, while for three-dimensional figures, axiom 1′ does not follow from axioms 1–3. Therefore, it is necessary to include this axiom in the definition—otherwise, surprising as it may seem, many functions different from the true volume will fit this definition (such functions arise from the *Dehn invariant*; see, for instance, the paper [3]).

Supplement B
Handout materials

One Lesson on Dimension (Simpler Version)

Problem 0. After seven washes, the length, width, and height of a bar of soap has been halved. How many more washes will the remaining bar take? (For each wash, the same amount of soap is used.)

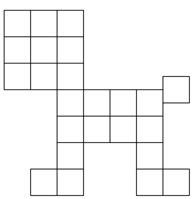
Problem 1. A loader in a warehouse can lift a package of $3 \times 3 \times 3$ one-litre cartons of milk. Will three loaders be able to lift a package of $9 \times 9 \times 9$ cartons?

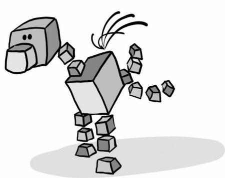
Problem 2. A children's inflatable pool is 30 cm deep, and its bottom is a square of side 1 m. What is the weight of such a pool filled with water?

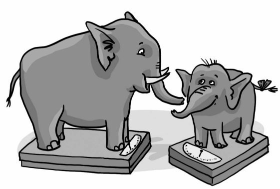
Problem 3. (a) Alex put together a picture consisting of squares of side 2 cm (see the picture on the left) and John, a similar picture of squares of side 4 cm. How many times is the area of Alex's picture smaller than John's?

(b) A cubic figure is made up of several wooden cubes (see the central picture). How will its mass change if the size of each cube is doubled?

(c) How will the mass of an elephant change if all of its dimensions are doubled? (Of course, you are not allowed to assume that the elephant has the shape of a parallelepiped.) How will the area of the elephant in the snapshot (see the picture on the left) change?







Problem 4. (a) (a) What fraction of the area of a square of size 12×12 is taken up by the border cells? (b) What fraction of the volume of a cube of size $12 \times 12 \times 12$ is taken up by the boundary bricks?

Problem 5. (b) A farm store sells two varieties of watermelons of the same diameter. A watermelon of the first variety costs 100 rubles but has a very thin peel. A watermelon of the second variety costs 70 rubles, but its peel takes up 20% of the diameter (and has to be thrown away). Which variety of watermelons is it more profitable to buy?

Problem 6. The length of the equator of a globe is 1 m. (a) What is its scale? (b) What is the area of Russia on this globe? (The length of the Earth's equator equals 40 000 km; the area of Russia is about 17 000 000 km^2.)

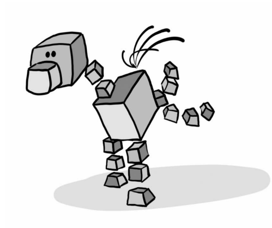
One Lesson on Dimension (Harder Version)

Problem 0. After seven washes, the length, width, and height of a bar of soap has been halved. How many more washes will the remaining bar take? (For each wash, the same amount of soap is used.)

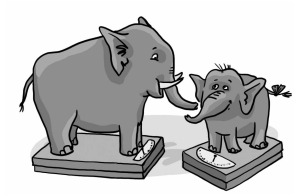
Problem 1. A loader in a warehouse can lift a package of $3 \times 3 \times 3$ one-litre cartons of milk. Will three loaders be able to lift a package of $9 \times 9 \times 9$ cartons?

Problem 2. A children's inflatable pool is 30 cm deep, and its bottom is a square of side 1 m. What is the weight of such a pool filled with water?

Problem 3. (a) A cubic figure is made up of several wooden cubes (see the picture). How will its mass change if the size of each cube is doubled?



(b) How will the mass of an elephant change if all of its dimensions are doubled?



Problem 4. (a) Denote the area of the circular disc of radius 1 by V_2. What is the area of the disc of radius R? (b) Denote the volume of the ball of radius 1 by V_3. What is the volume of a ball of radius R?

Problem 5. Two balls of radii 3 and 5 were placed on the left pan of the scales and one ball of radius 8, on the right pan. Which of the pans will outweigh the other? (All balls are made of the same material.)

Problem 6. (a) What fraction of the volume of a cube of size $12 \times 12 \times 12$ is taken up by the boundary bricks?

(b) A supermarket sells two varieties of watermelons of the same diameter. A watermelon of the first variety costs 100 rubles, but has very thin peel. A watermelon of the second variety costs 70 rubles, but its peel takes up 20% of diameter (and has to be thrown away). Which variety of watermelons is it more profitable to buy?

Problem 7. The length of the equator of a globe equals 1 m. (a) What is the scale of the globe? (b) What is the area of Russia on this globe? (The length of the Earth's equator equals 40 000 km; the area of Russia is about 17 000 000 km^2.)

Additional Problems

Problem 8. The width of a flat copper ring increases 1.5 times when the ring is heated. How does the area of the hole change?

Problem 9*. The Earth was hooped along the equator. Then the hoop was lengthened by 1 m (so that the arising gap was the same everywhere). Can a cat slip under the hoop?

Problem 10*. A balloon (having the shape of an ideal ball) was blown up so that its area increased by 9%. What happened to its radius?

Problem 11*. Is it possible to cut out several discs from a square of side 10 cm and put them in line so as to obtain a chain longer than one kilometre?

Problem 12*. Two round coins were put on the left pan of the scales and another one was put on the right pan, and the scales were in balance. Which of the pans will outweigh the other if each of the coins is replaced by a ball of the same radius? (All the balls and coins are made of the same material, and all the coins have the same thickness.)

Two Lessons on Dimension (First Lesson)

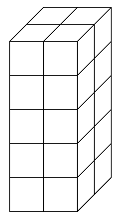
Problem 0. After seven washes, the length, width, and height of a bar of soap has been halved. How many more washes will the remaining bar take? (For each wash, the same amount of soap is used.)

Problem 1. A square of side (a) 3 cm; (b) 1 m was cut into squares of side 1 cm. How many squares were obtained? A cube of side (c) 3 cm; (d) 1 m was cut into cubes of side 1 cm. How many small cubes were obtained?

Problem 2. A loader in a warehouse can lift a package of $3 \times 3 \times 3$ one-litre cartons of milk. Will three loaders be able to lift a package of $9 \times 9 \times 9$ cartons?

Problem 3. A children's inflatable pool is 30 cm deep, and its bottom is a square of side 1 m. What is the weight of such a pool filled with water?

Problem 4. Alex and John each built a tower of cubes. Both towers have a square base and consist of the same number of cubes.

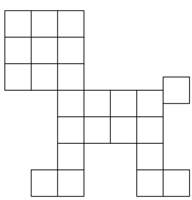


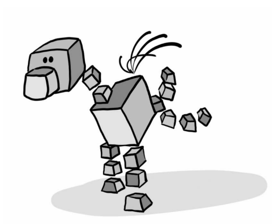
(a) The side of the base of John's tower is four times longer than Alex's. How many times higher is Alex's tower than John's?

(b) Alex's tower is four times higher than John's. How many times longer is the side of the base of John's tower than that of Alex's tower?

Problem 5. (a) Alex put together a picture consisting of squares of side 2 cm (see the picture on the left), and John, a similar picture of squares of side 4 cm. How many times is the area of Alex's picture smaller than John's?

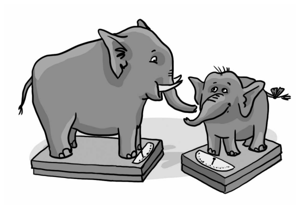
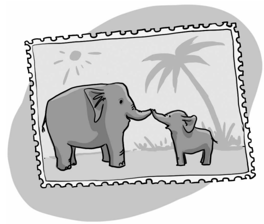
(b) A cubic figure is made up of several wooden cubes (see the picture on the right). How will its mass change if the size of each cube is doubled?





Problem 6. How will the mass of an elephant change if all of its dimensions are doubled? (Of course, you are not allowed to assume that

an elephant has the shape of a parallelepiped.) How will the area of the elephant in the snapshot change?

Problem 7. (a) Denote the area of a circular disk of radius 1 by V_2. What is the area of a disk of radius R? (b) Denote the volume of a ball of radius 1 by V_3. What is the volume of a ball of radius R?

Two Lessons on Dimension (Second Lesson)

Problem 0. What is the surface area of a cube of side (a) 10 cm; (b) 12 cm?

Problem 1. A cube of side 1 cm is cut into cubic bricks of side 1 cm from which a tower with base of side 1 is built. What is the area of the tower's surface? Is it smaller or greater than the surface area of the initial cube? How many times?

Problem 2. A cube of side 12 built from small cube bricks of side 1 is painted white. How many bricks have no faces painted; one face painted; two faces painted; three faces painted?

Problem 3. (a) What fraction of the area of a square of size 12×12 is taken up by the border cells?

(b) What fraction of the volume of a cube of size $12 \times 12 \times 12$ is taken up by the boundary bricks?

Problem 4. A supermarket sells two varieties of watermelons of the same diameter. A watermelon of the first variety costs 100 rubles, but has a very thin peel. A watermelon of the second variety costs 70 rubles, but its peel takes up 20% of the diameter (and has to be thrown away). Which variety of watermelons is it more profitable to buy?

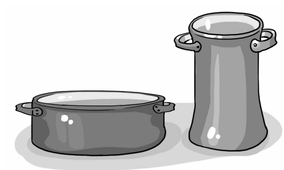
Problem 5. The length of the equator of a globe equals 1 m. (a) What is the scale of the globe? (b) What is the area of Russia on this globe? (The length of the Earth's equator is 40 000 km; the area of Russia is about 17 000 000 km^2.)

Problem 6. A paper belt is put on a cube of size (a) $3 \times 3 \times 3$; (b) $100 \times 100 \times 100$ as tightly as possible. What clearance will result from lengthening the belt by 8 (so that the belt will remain square)?

Problem 7. The earth was hooped along the equator. Then the hoop was lengthened by 1 m (so that the arising gap was everywhere the same). Can a cat slip under the hoop?

Additional Problems

Problem 8. Which of the pots can be filled with the most amount of liquid, the left one, which is wider, or the right one, which is three times higher, but twice narrower?



Problem 9. Two balls of radii 3 and 5 were placed on the left pan of the scales and one ball of radius 8 on the right pan. Which of the pans will outweigh the other? (All balls are made of the same material.)

Problem 10*. Two round coins were put on the left pan of the scales and another one was put on the right pan, and the scales were in balance. Which of the pans will outweigh the other if each of the coins is replaced by a ball of the same radius? (All the balls and coins are made of the same material, and all the coins have the same thickness.)

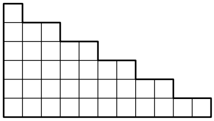
Problem 11. Is it possible to cut out several discs from a square of side 10 cm and put them in line so as to obtain a chain longer than one kilometre?

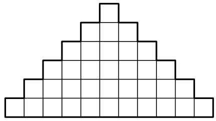
Problem 12. The width of a flat copper ring increases 1.5 times when the ring is heated. How does the area of the hole change?

Problem 13. A balloon (having the shape of an ideal ball) was blown up so that its area increased by 9%. What happened to its radius?

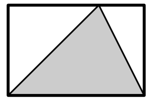
A Lesson on Areas and Sums

Problem 1. (a) Which of the figures in the pictures contains more squares? (b) Count these squares.

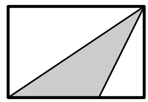




Problem 2. A triangle lies in a rectangular box, so that one of its sides coincides with the bottom edge of the box and the remaining vertex lies on the opposite edge (see the picture on the right). What fraction of the area of the box does the triangle occupy?

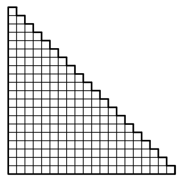


Problem 3. (a) Another triangle was placed sideways in a box (see the picture on the right). Is the fraction of the box area occupied by it greater than, smaller than, or equal to that in the previous problem?



(b)* Does a triangle of area 10 fit in a box of area 19?

Problem 4. The area of an isosceles right triangle is half the area of a square whose side is equal to the leg of this triangle. What is the area of a "pixel" (composed of unit squares) isosceles right triangle with leg, say, 20 (see the picture on the right)?



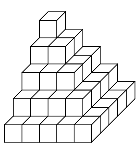
Definition. The area of a pixel triangle with leg n (that is, the number $1+2+3+\ldots+n$) is called the nth *triangular number* and denoted by T_n.

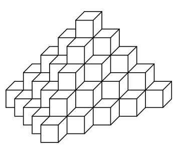
Problem 5. Find $T_{100} = 1+2+3+\ldots+100$.

Problem 6. Find the sum of all two-digit numbers divisible by 7.

Problem 7. Find the sum of two consecutive triangular numbers.

Problem 8. Which of the figures contains more cubes?

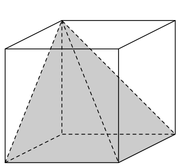


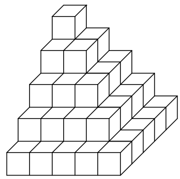


Additional Problems

Problem 9. Prove the addition theorem for triangular numbers: $T_{n+m} = T_n + T_m + nm$.

Problem 10. What part of a cubic box is taken up by the (irregular) quadrangular pyramid (see the picture on the left)?





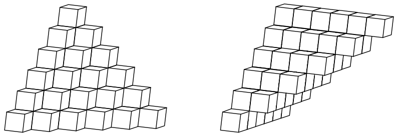
Problem 11*. (a) In the corner of a room a pyramid of height n is stacked (see the picture on the right). How many cubes does it contain? (b) Calculate the sum $1^2 + \ldots + n^2$.

Problem 12*. Find the nth *pyramidal number*, that is, find the sum of $T_1 + T_2 + \ldots + T_n$ consecutive triangular numbers.

Problem 13*. Prove geometrically that $1 + \ldots + n^3 = (1 + \ldots + n)^2$.

Lesson on Cavalieri's Principle

Problem 0. Which of the pyramids shown in the figures contains more cubes?



Given two solids in space, suppose that all planes parallel to a fixed one are drawn. *Cavalieri's principle* asserts that if the area of the section of the first solid by each plane is equal to the area of the section of the second solid by the same plane, then the volumes of the solids are equal.

Problem 1. Formulate an analogue of Cavalieri's principle for planar figures.

Problem 2. Prove Cavalieri's principle (a) for trapezoids with bases parallel to the sections; (b) for convex polygons in the plane.

Problem 3. A solid is expanded by a factor of k in one direction. What happens to its volume?

Definition. In this lesson, by a *cone* we mean a body consisting of a planar figure ('the base of the cone') together with all straight line segments joining it to a fixed point ('the vertex, or apex, of the cone') that lies outside the plane of the base.

Problem 4. Suppose that a cone has a base of area S and its altitude is h. Find the area of the section of this cone by a plane parallel to the base and located at a distance of x from the vertex.

Problem 5. Prove that the volume of a cone depends only on its altitude and on the area of its base (and does not depend on the shape of the base).

Problem 6. Given that the volume of a cone of altitude 1 with base of area 1 equals c, calculate the volume of a cone of height h with base of area S.

Problem 7. Considering a pyramid with square base, find c.

Problem 8. Find the area of the section of a ball of radius R by a plane located at the distance x from the centre of the ball.

Additional Problems

Problem 9*. Prove that

$$T_1 + T_2 + \ldots + T_n = n \cdot 1 + (n-1) \cdot 2 + (n-2) \cdot 3 + \ldots + 1 \cdot n.$$

Problem 10*. Prove that the ratio of the volumes of the ball and of a polyhedron circumscribed about it is equal to that of the surface areas of thes ball and the polyhedron.

Problem 11*. (a) On square lined paper, a polygon M whose vertices are at grid points and whose sides do not go along grid lines is sketched. Prove that the sum of lengths of vertical segments of the grid lines located inside M is equal to the sum of lengths of horizontal segments of the grid lines inside M.

(b) What is a generalisation of this statement to polygons whose sides are allowed to go along grid lines?

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[1] I.V. Artamkin, A.L. Gorodentsev, A.G. Kulakov, M.A. Prokhorov, S.M. Khoroshkin, A.V. Khokhlov. *Numbers and sums* // Mathematical Education, № 2-3 (9-10), April-September 1999, pp. 2-57.

Preprint available at
http://gorod.bogomolov-lab.ru/ps/scho/ns.ps.gz.

[2] Ya.A. Perelman. *Amusing Geometry*, Moscow-Leningrad GTTI, 1950.

Chapter 11 contains a discussion of problems about dimensions in the spirit of the first two lessons of the present book. Perelman's book is available at
https://math.ru/lib/book/djvu/perelman/zanim_geom.djvu

[3] D.B. Fuchs. *Can One Make a Tetrahedron out of a Cube?* // Kvant, 1990, № 11.

Contains an accessible exposition of the Dehn invariant, which explains why Cavalieri's Principle (or an equivalent axiom) must be included in the definition of the volume of a polyhedron. The article is available at
http://kvant.mccme.ru/1990/11/mozhno_li_iz_tetraedra_sdelat.htm

[4] I.V. Yashchenko. *Paradoxes in set theory*, Moscow, MCCME Publs., 2002.

Explains why it is impossible to define area for all planar geometric figures. The booklet is available at

https://mccme.ru/free-books/mmmf-lectures/book.20.pdf

[4] A.D. Blinkov. *The computation of certain finite sums* // Mathematics for school children, № 4, 2008.

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